Generalizing abstract model theory, with an eye toward applications

(Joint with Jiří Rosický)

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My goal is to give a resolutely nontechnical talk: a (hopefully) compelling story leading from classical model theory to the category-theoretic analysis of abstract model theory.

1. Classical (first-order) model theory.
2. First-order problems, infinitary solutions: generalized logics.
3. Logic-independence: abstract elementary classes (AECs).
4. From discrete to continuous: metric AECs.
5. A unifying category-theoretic framework.

Each step in this progression is driven, as we will see, by concrete, genuinely mathematical considerations.

Modus Operandi

Given a category of interesting mathematical structures $\mathcal{K}$, we:

- Identify the vocabulary L needed to capture their structure.
- Find a set of (first-order) sentences that characterize the objects in $\mathcal{K}$—its *theory* T.
- Restrict to suitably structure-preserving mappings—ideally characterized by preservation of a class of logical formulas.

Successive refinements:

$$\mathbf{Elem}(T) \hookrightarrow \mathbf{Mod}(T) \hookrightarrow \mathbf{Str}(L)$$

If we've done our job well, and our luck is good, we will obtain $\mathbf{Elem}(T) = \mathcal{K}$.

One of the most fundamental (and awesome) properties of first order logic: it's "compact."

Theorem (Compactness Theorem)

Version 1: Let Γ *be an infinite set of first order sentences. If* Γ *is inconsistent, then there is a finite set of sentences* $\Gamma' \subset \Gamma$ *that is itself inconsistent.*

Version 2: Let Γ *be an infinite set of first order sentences. If for any finite* $\Gamma' \subset \Gamma$ *there is an object* $X_{\Gamma'}$ *obeying all of the sentences in* Γ'*, then there is a single object that obeys the entire infinite list* Γ*.*

The second version makes clear: compactness is a magic trick, which allows us to produce structures to exact specifications.

There's a price to pay, though.

Consider the natural numbers with successor: $\langle \mathbb{N}, 0, S \rangle$.

Here $L = \langle 0, S \rangle$, and T contains, e.g.

$$\neg \exists x(0 = Sx) \quad \text{and} \quad \forall x(x \neq 0 \rightarrow \exists y[x = Sy])$$

In fact we take T to be the *complete theory* of $\langle \mathbb{N}, 0, S \rangle$, the set of all first-order sentences in 0 and S that it obeys.

Surely there is only one object that satisfies T, namely $\mathbb{N}$ itself. At the very least, $\mathbb{N}$ must be the only countable object, right?

Strange models: Expand the vocabulary by a new constant symbol c, and consider the set of sentences

$$\Gamma = T \cup \{c \neq 0, c \neq S0, c \neq SS0, c \neq SSS0, \ldots\}.$$

By compactness, there is some $\mathfrak{N}$ that obeys all of the sentences simultaneously: it contains some element named by c that is neither 0 nor a successor. That is, $\mathfrak{N}$ contains a *nonstandard natural number*! It gets worse, too...

Punchline: there are lots and lots of nonisomorphic countable versions of the natural numbers with successor, and first-order logic is incapable of distinguishing between them.

Consider the good old-fashioned real numbers:

$$\langle \mathbb{R}, 0, 1, \times, +, \leq \rangle$$

Its complete first-order theory is T_{RCF}, which is incredibly nice—o-minimal, among other things.

There's a great deal of research on expansions by new functions:

$$\langle \mathbb{R}, 0, 1, \times, +, \leq, f \rangle$$

Under what assumptions on the interpretation of f is the structure still nice, e.g. o-minimal?

Some expansions are definitely a problem, though:

$$\langle \mathbb{R}, 0, 1, \times, +, \leq, \sin(\pi -) \rangle$$

Here the natural numbers are definable, by the formula

$$\sin(\pi x) = 0 \wedge x \geq 0$$

Thus our models may conceal terrifying monsters. The prevailing solution here is to simply restrict the domain.

A more telling example:

$$\langle \mathbb{C}, 0, 1, \times, + \rangle$$

Its complete first-order theory is T_{ACF_0}, which is even nicer—strongly minimal!

And yet you would also want the exponential function:

$$\langle \mathbb{C}, 0, 1, \times, +, \exp(-) \rangle$$

There's the same terrible price to pay. But what if we could force the natural numbers or, say, the integers, to be exactly what they should be?

$$\text{Hrushovski: } \forall x(\exp(x) = 1 \to \bigvee_{n \in Z} x = 2n\pi)$$

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- $L_{\kappa,\lambda}$, where we are permitted
  1. conjunctions/disjunctions of $< \kappa$ formulas, and
  2. quantification over $< \lambda$ variables.
- $L(Q)$, where Q is the counting quantifier "there exist uncountable many."
- $L_{\kappa,\lambda}(Q)$.

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These strenuously resist any uniform treatment.

So we may need to work more generally in the first step of the process of refinement described at the start:

$$\mathbf{Elem}(T) \hookrightarrow \mathbf{Mod}(T) \hookrightarrow \mathbf{Str}(L)$$

Whether or not a first-order T is suitable, we still need to think about the morphisms. Sometimes these are essentially syntactic...

Example

Let T_{Ab} be the first order theory of Abelian groups.

- Injective homomorphisms: preserve quantifier free formulas.
- Pure embeddings: preserve positive primitive formulas.
- Elementary embeddings: preserve all first-order formulas.

Example (Baldwin/Eklof/Trlifaj)

For N an Abelian group, define

$$^{\perp}N = \{A \,|\, \mathrm{Ext}^i(A, N) = 0 \text{ for } 1 \leq i < \omega\}$$

with morphisms the injective maps $f : A \to B$ with $B/f(A) \in {}^{\perp}N$.

This is a mathematically natural category, but the morphisms are grotesquely nonlogical. So we need to be more flexible with those as well.

Something magical happens if N is cotorsion, by the way...

Modus Operandi

Avoid any reliance on an ambient logic, and find a purely semantic characterization of subcategories of structures

$$\mathcal{K} \quad \hookrightarrow \quad \mathbf{Str}(L)$$

that are—morally speaking—generalized categories of models, simultaneously

- general enough to subsume the kinds of examples already considered,
- but with enough structure to support a robust array of old and new model-theoretic techniques.

This approach is due to Shelah, as is the next concept in our progression.

Definition

Let L be a finitary vocabulary. An abstract elementary class (or AEC) in L consists of a class $\mathcal{K}$ of L-structures together with a strong substructure relation $\prec_{\mathcal{K}}$ with properties that include:

- *Tarski-Vaught: $\mathcal{K}$ is closed under unions of $\prec_{\mathcal{K}}$-chains.*
- *Coherence: If $M \subseteq N \prec_{\mathcal{K}} M'$ and $M \prec_{\mathcal{K}} M'$, then $M \prec_{\mathcal{K}} N$.*
- *Löwenheim-Skolem: There is an infinite cardinal $LS(\mathcal{K})$ such that for any $M \in \mathcal{K}$ and subset A of M, there is $N \in \mathcal{K}$ with $A \subseteq N \prec_{\mathcal{K}} M$, and $|N| \leq |A| + LS(\mathcal{K})$.*

The $\mathcal{K}$-morphisms are injective maps $f : M \to N$ with $f[M] \prec_{\mathcal{K}} N$.

Coherence tells us about the way $\mathcal{K}$ sits inside $\mathbf{Str}(L)$. The other two axioms tell us about $\mathcal{K}$ as an abstract category:

- Tarski-Vaught: $\mathcal{K}$ has directed colimits (i.e. direct limits), and these colimits are concrete.
- Löwenheim-Skolem: Any M in $\mathcal{K}$ can be built as a highly directed colimit of structures of size $\mathrm{LS}(\mathcal{K})$.

This will be significant later.

Examples

1. Abelian groups and pure embeddings form an AEC.
2. The Ext-orthogonality class of Abelian groups $^{\perp}N$ forms an AEC when N is cotorsion (Baldwin/Eklof/Trlifaj).
3. Elementary classes of models are AECs.
4. Classes of models in the generalized logics above form AECs under suitable assumptions on $\prec_{\mathcal{K}}$.

So they are in fact very general, but not too general: there is a vast—and constantly expanding—literature on their classification theory.

The question of stability has a little added mathematical resonance in AECs.

As there is no ambient logic, types are not syntactic but algebraic: we speak of *Galois types* over models $M \in \mathcal{K}$, which we identify with orbits in a large “monster” model under automorphisms that fix M.

Baldwin/Eklof/Trlifaj offer a very pleasant characterization of Galois types in ${}^{\perp}N$, and a discussion of stability in that setting.

Metric abstract elementary classes (mAECs) are a recent development (due to Hirvonen/Hyttinen) in the project to develop a model theory relevant to structures arising in analysis, e.g. Banach spaces.

Slogan

Metric AECs represent an amalgam of AECs and the program of continuous logic.

Roughly, an mAEC is an AEC whose structures are built on complete metric spaces, rather than discrete sets.

A few crucial changes to the axioms for an mAEC $\mathcal{K}$:
(1) In the Löwenheim-Skolem axiom, cardinality is replaced by *density character*,

$$\mathrm{dc}(M) = \min\{|X| \mid X \text{ is a dense subset of } M\}$$

Upshot: the crucial notion of size in an mAEC is density character, not cardinality:

- $\mathcal{K}$ is λ-*d-categorical* if it contains one model of density character λ up to iso.
- $\mathcal{K}$ is λ-*d-stable* if any model of density character λ has Galois type space of density character at most λ.

(2) While the union of an increasing chain may not belong to an mAEC $\mathcal{K}$, the completion of the union must. Upshot:

- $\mathcal{K}$ is closed under colimits of chains, hence under arbitrary directed colimits.
- These colimits need not be concrete: if $U : \mathcal{K} \to \mathbf{Sets}$ is the forgetful functor, in general we may have

$$U(\mathrm{colim}_{i\in I} M_i) \supsetneq \mathrm{colim}_{i\in I} UM_i$$

That is, U will not preserve directed colimits...

Fact

$\aleph_1$*-directed colimits are concrete!*

Examples

- Any AEC is an mAEC.
- Hilbert spaces with a unitary operator (Argoty/Berenstein).
- Probability spaces with an automorphism (Berenstein/Henson).
- Gelfand triples (Zambrano).

Whether $\mathcal{K}$ is an AEC or an mAEC, it can be built via colimits of a set of small objects—it's an *accessible category*—with arbitrary directed colimits.

What differs is the level of concreteness of the colimits involved: directed colimits are concrete in AECs, $\aleph_1$-directed are concrete in mAECs...

Big picture:

We give a uniform treatment of AECs and mAECs as pairs

$$\mathcal{K}, U : \mathcal{K} \to \textbf{Sets}$$

with $\mathcal{K}$ an accessible category with all directed colimits and all morphisms monomorphisms, and U a functor whose properties can be tuned to the desired model theoretic frequency.

Analysis of $\mathcal{K}$ as an abstract category allows uniform treatment of

- Presentation theorems
- Ehrenfeucht-Mostowski functors, $E : \textbf{Lin} \to \mathcal{K}$

Adjusting U allows us to capture subtle differences in concreteness/discreteness...

Theorem (L/Rosický)

Abstract elementary classes are precisely the pairs $(\mathcal{K}, U)$, with U a functor from $\mathcal{K}$ to **Sets**, *where*

- $\mathcal{K}$ *is accessible,* **has all directed colimits**, *and all morphisms are monomorphisms.*
- U *is faithful, coherent, and preserves monomorphisms and* **directed colimits**.
- $(\mathcal{K}, U)$ *is replete and iso-full...*

Theorem (L/Rosický)

Let $\mathcal{K}$ be an mAEC, with $U : \mathcal{K} \to$ **Sets** *the forgetful functor.*

- *$\mathcal{K}$ is accessible,* **has all directed colimits**, *and all morphisms are monomorphisms.*
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So at the beating heart of each is an accessible category with directed colimits—this allows the promised uniform treatment of AECs and mAECs.

Proposition (Beke/Rosický)

In any accessible category $\mathcal{K}$ that has all directed colimits, each object M has a well-defined internal size in $\mathcal{K}$, denoted $|M|_{\mathcal{K}}$.

This size ends up meaning precisely what we would like:

Note

- *If $\mathcal{K}$ is an AEC then for any $M \in \mathcal{K}$, $|M|_{\mathcal{K}} = |M|$.*
- *If $\mathcal{K}$ is an mAEC then for any $M \in \mathcal{K}$, $|M|_{\mathcal{K}} = dc(M)$.*

So, in fact, we end up with the appropriate notions of size by default.

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Classical construction: inflate a set of indiscernibles indexed by I, closing it under Skolem functions—a wildly syntactic affair. In the abstract context, this first requires a reintroduction of syntax, then painful checking, e.g. whether Skolem functions are continuous.

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We escape this completely...

Theorem (L/Rosický)

If $\mathcal{K}$ is a large accessible category with directed colimits and all morphisms mono, it admits an EM-functor

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Because E preserves directed colimits, it eventually preserves (internal) sizes: for sufficiently large I in **Lin**,

$$|EI|_{\mathcal{K}} = |I|$$

This, and simple functoriality of E, are surprisingly powerful, and lead to dramatic new results on stability in mAECs.

Definition (L/Rosický)

A μ-concrete AEC, or μ-CAEC, consists of a pair $(\mathcal{K}, U)$, where $U : \mathcal{K} \to$ **Sets** and

- $\mathcal{K}$ is accessible, has all directed colimits, and all morphisms are monomorphisms.
- U is faithful, coherent, and preserves monomorphisms and μ-**directed colimits**.
- $\mathcal{K}$ is iso-full.

This is one of many possible generalized frameworks for abstract model theory that have popped up recently.

Another possible route, μ-AECs, drops the assumption of closure under directed colimits (Boney/Grossberg/L/Rosický/Vasey). This is costly, but encompasses:

- AECs and mAECs, of course.
- μ-complete boolean algebras.
- Classes of μ-saturated objects.

As it happens,

Theorem

The μ-AECs are, up to equivalence of categories, precisely the accessible categories with all morphisms mono.

Coda

We've done something slightly funny in our analysis of mAECs: an extra act of forgetting.

$$\mathcal{K} \to \mathbf{Met} \to \mathbf{Sets}$$

This "discretization" loses us structure, clearly, and the ability to analyze, e.g. μ-d-tameness.
Perhaps we could (should?) have stuck with

$$\mathcal{K} \stackrel{U}{\to} \mathbf{Met}$$

Question: How much meaningful theory can we develop in this way?

Coda

A bigger question: Let $\mathcal{K}$ be accessible with directed colimits, monomorphisms.

- AECs: abstract model theory in sense of **Sets**,

$$\mathcal{K} \stackrel{U}{\to} \textbf{Sets}$$

- mAECs: abstract model theory in sense of **Met**,

$$\mathcal{K} \stackrel{U}{\to} \textbf{Met}$$

- Abstract model theory in sense of a general accessible category with directed colimits, $\mathcal{A}$,

$$\mathcal{K} \stackrel{U}{\to} \mathcal{A}?$$