Reconstructing functions and unique identification minors

Erkko Lehtonen

Institut für Algebra
Technische Universität Dresden
Erkko.Lehtonen@tu-dresden.de

joint work with
Miguel Couceiro (LORIA)
Karsten Schölzel (University of Luxembourg)

91st Workshop on General Algebra (AAA91)
Brno, 5–7 February 2016

Minors

$f\colon A^n \to B, \quad g\colon A^m \to B$

f is a **minor** of g (denoted $f \leq g$)
if there exists $\sigma\colon \{1, \dots, m\} \to \{1, \dots, n\}$ such that

$$f(a_1, \dots, a_n) = g(a_{\sigma(1)}, \dots, a_{\sigma(m)})$$

for all $a_1, \dots, a_n \in A$.

The minor relation $\leq$ is a quasiorder on $\mathcal{F}_{AB}$.

f and g are **equivalent** (denoted $f \equiv g$)
if $f \leq g$ and $g \leq f$.

Identification minors

$\binom{n}{2}$ = the set of all 2-element subsets of $\{1, \ldots, n\}$

$f\colon A^n \to B \quad (n \geq 2)$

For $I = \{i, j\} \in \binom{n}{2}$ with $i < j$, define $f_I\colon A^{n-1} \to B$ as

$$f_I(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_{j-1}, a_i, a_j, \ldots, a_{n-1}),$$

for all $a_1, \ldots, a_{n-1} \in A$.

f_I is an **identification minor** of f.

Identification minors – examples

Example

Let $f\colon \mathbb{R}^3 \to \mathbb{R}$,

$$f(x_1, x_2, x_3) = x_1^3 - x_2^2 x_3.$$

The identification minors of f are the following:

$$f_{\{1,2\}} = x_1^3 - x_1^2 x_2, \qquad f_{\{1,3\}} = x_1^3 - x_1 x_2^2, \qquad f_{\{2,3\}} = 0.$$

Identification minors – examples

Example

Let $n \geq 2$ and let $f\colon \{0,1\}^n \to \{0,1\}$ be given by the rule

$$f(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$$

(addition modulo 2).

For every $I \in \binom{n}{2}$, f_I is equivalent to the function $g\colon \{0,1\}^{n-1} \to \{0,1\}$,

$$g(x_1, \ldots, x_{n-1}) = x_1 + \cdots + x_{n-2}.$$

Reconstruction problem for functions

Question

Is a function $f\colon A^n \to B$ uniquely determined, up to equivalence, by the collection of its identification minors?

Reconstruction problem for functions

Question

Is a function $f\colon A^n \to B$ uniquely determined, up to equivalence, by the collection of its identification minors?

Example

$$f\colon \{0,1\}^3 \to \{0,1\}, \quad f(x_1, x_2, x_3) = x_1 + x_2 + x_3$$
$$g\colon \{0,1\}^3 \to \{0,1\}, \quad g(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$$
$$h\colon \{0,1\}^3 \to \{0,1\}, \quad h(x_1, x_2, x_3) = x_1$$

(addition and multiplication modulo 2)

All identification minors of f, g, and h are projections.

Reconstruction problem for functions

Assume that $n \geq 2$ and $f\colon A^n \to B$.

1. The **deck** of f, denoted $\operatorname{deck} f$, is the multiset $\langle f_I/{\equiv} : I \in \binom{n}{2} \rangle$. The elements of the deck of f are called **cards** of f.
2. A function $g\colon A^n \to B$ is a **reconstruction** of f, if $\operatorname{deck} f = \operatorname{deck} g$.
3. A function is **reconstructible** if it is equivalent to all of its reconstructions.
4. A class $\mathcal{C} \subseteq \mathcal{F}_{AB}$ of functions is **reconstructible**, if all members of $\mathcal{C}$ are reconstructible.
5. A class $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is **weakly reconstructible**, if for every $f \in \mathcal{C}$, all reconstructions of f that are members of $\mathcal{C}$ are equivalent to f.
6. A class $\mathcal{C} \subseteq \mathcal{F}_{AB}$ is **recognizable**, if all reconstructions of members of $\mathcal{C}$ are members of $\mathcal{C}$.

Reconstruction problem for functions

Reconstructible classes:

- totally symmetric functions
- affine functions over finite fields

Weakly reconstructible classes:

- affine functions over cancellative nonassociative right semirings
- linear functions over nonassociative right semirings
- functions determined by the order of first occurrence

Negative results:

- Infinite families of non-reconstructible monotone functions were discovered in
  M. COUCEIRO, E. LEHTONEN, K. SCHÖLZEL, Hypomorphic Sperner systems and non-reconstructible functions, *Order* **32** (2015) 255–292.

Reconstruction problem for functions

As a tool, we formulated and solved reconstruction problems for other kinds of mathematical objects.

For example, linear functions

$$f(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

are completely described, up to permutation of arguments, by the multiset $\langle a_1, a_2, \ldots, a_n \rangle$ of coefficients.

Functions with a unique identification minor

A function $f\colon A^n \to B$ has a **unique identification minor**, if $f_I \equiv f_J$ for all $I, J \in \binom{n}{2}$.

Problem

Determine all functions with a unique identification minor.

This problem was previously posed in
M. Bouaziz, M. Couceiro, M. Pouzet, Join-irreducible Boolean functions, *Order* **27** (2010) 261–282.

Functions with a unique identification minor

A function $f\colon A^n \to B$ has a **unique identification minor**, if $f_I \equiv f_J$ for all $I, J \in \binom{n}{2}$.

Problem

Determine all functions with a unique identification minor.

This problem was previously posed in
M. Bouaziz, M. Couceiro, M. Pouzet, Join-irreducible Boolean functions, *Order* **27** (2010) 261–282.

Functions with a unique identification minor

Example

Examples of functions with a unique identification minor:

- 2-set-transitive functions,
- functions determined by the order of first occurrence,
- functions determined by content and singletons,
- other known examples of small arity.

2-set-transitive functions

$f\colon A^n \to B$, $\quad \sigma \in S_n$

f is **invariant** under σ if for all $a_1, \dots, a_n \in A$,

$$f(a_1, \dots, a_n) = f(a_{\sigma(1)}, \dots, a_{\sigma(n)}).$$

The set of all permutations under which f is invariant constitutes a permutation group, the **invariance group** of f, and it is denoted by $\operatorname{Inv} f$.

f is **totally symmetric** if $\operatorname{Inv} f = S_n$.

A permutation group G is 2**-set-transitive** if for all $I, J \in \binom{n}{2}$, there exists a permutation $\sigma \in G$ such that $\sigma[I] = J$.

f is 2**-set-transitive** if $\operatorname{Inv} f$ is 2-set-transitive.

Functions determined by the order of first occurrence

$\mathrm{ofo}\colon A^* \to A^\sharp$

ofo(**a**) is the string obtained from **a** by removing all repeated occurrences of elements, retaining only the first occurrence of each element occurring in **a**

Example

ofo(miscalculations) $=$ miscaluton

ofo(unprosperousness) $=$ unprose

ofo(exclusion) $=$ exclusion

$f\colon A^n \to B$ is **determined by the order of first occurrence** if there exists a map $f^*\colon A^\sharp \to B$ such that $f = f^* \circ \mathrm{ofo}|_{A^n}$.

Functions determined by content and singletons

$\mathrm{cs}\colon A^* \to \mathcal{M}(A) \times A^\sharp$
$\mathrm{cs}(\mathbf{a}) = (\mathrm{ms}(\mathbf{a}), \mathrm{sng}(\mathbf{a}))$
$\mathrm{ms}(a_1, \dots, a_n) = \langle a_1, \dots, a_n \rangle$
$\mathrm{sng}(\mathbf{a})$ lists the letters occurring in $\mathbf{a}$ exactly once, in the order of appearance

Example

$\mathrm{cs}(\mathrm{miscalculations}) = (\langle \mathrm{a}^2, \mathrm{c}^2, \mathrm{i}^2, \mathrm{l}^2, \mathrm{m}, \mathrm{n}, \mathrm{o}, \mathrm{s}^2, \mathrm{t}, \mathrm{u} \rangle, \mathrm{muton})$

$\mathrm{cs}(\mathrm{unprosperousness}) = (\langle \mathrm{e}^2, \mathrm{n}^2, \mathrm{o}^2, \mathrm{p}^2, \mathrm{r}^2, \mathrm{s}^4, \mathrm{u}^2 \rangle, \varepsilon)$

$\mathrm{cs}(\mathrm{exclusion}) = (\langle \mathrm{c}, \mathrm{e}, \mathrm{i}, \mathrm{l}, \mathrm{n}, \mathrm{o}, \mathrm{s}, \mathrm{u}, \mathrm{x} \rangle, \mathrm{exclusion})$

$f\colon A^n \to B$ is **determined by content and singletons** if there exists a map $f^*\colon \mathcal{M}(A) \times A^\sharp \to B$ such that $f = f^* \circ \mathrm{cs}|_{A^n}$.

Current work and open problems

- What are the functions with a unique identification minor?
- Are the functions with a unique identification minor reconstructible?
- Is the class of functions with inessential arguments reconstructible?
- How about set-reconstructibility?
- How about reconstructibility from a few cards?
- …

Thank you.