GRADED PRIME IDEALS

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Introduction

Definition:

Let G be a group with identity e. Then a ring R is a G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$.

The elements of R_g are called homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $x \in R$, then x can be written uniquely as $\Sigma_{g\in G} x_g$, where x_g is called homogeneous component of x in R_g. Moreover, R_e is a subring of R and $1 \in R_e$.

Introduction

Example Let R be any ring and G be any group with identity e. Then R is G-graded by $R_e = R$ and $R_g = 0$ for all $g \in G\text{-}\{e\}$. This graduation is called the trivial graduation of R by G.

Example

Let $R = K[x]$, where K is a field, and $G = Z$. Then R is G-graded by
$R_0 = K$, $R_i = Kx^i$ for $i > 0$ and $R_i = 0$ for $i < 0$. This is called the usual graduation of $K[x]$ by Z.

Definition

Let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring . An ideal I of R is said to be a graded ideal if $I = \oplus_{g \in G} (I \cap R_g) := \oplus_{g \in G} I_g$. Thus if $x \in I$, then $x = \Sigma_{g \in G} x_g$ with $x_g \in I$.

The following example shows that an ideal of a G-graded ring need not be a graded ideal in general.

Example $R = \mathbb{Z}[i]$ (the Gaussian integers) and let $G = \mathbb{Z}_2$. Then R is G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Let I be the ideal of R generated by $x = (1 + i)$. Then $x_0 = 1$ and $x_1 = i$. Clearly, $x \in I$ while $x_0 \notin I$ because if $x_0 \in I$, then there exists $a + ib \in \mathbb{Z}[i]$ such that $1 = (a + ib)(1 + i)$, which implies $a - b = 1$ and $a + b = 0$. Hence $2a = 1$, a contradiction Thus I is not a graded ideal of R.

The concept of graded prime ideal was introduce in[6]. as a generalization of the notion of prime ideal.

Definition :
Let R be a G-graded ring. A proper graded ideal I of R is said to be graded prime ideal of R if whenever a and b are homogenous element of R such that $ab \in I$, then either $a \in I$ or $b \in I$.

Example Let $R = \mathbb{Z}[i]$ (the Gaussian integers) and let $G = \mathbb{Z}_2$. Then R is G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Let $I = 2R$. Then I is graded prime ideal which is not a prime ideal since $(1 + i)(1 - i) \in I$, $1 + i \notin I$ and $1 - i \notin I$.

Theorem Let R be a G –graded ring and I be a graded ideal of R. Then I is a graded prime ideal if and only if whenever J_1, J_2 are graded ideals of R with $J_1J_2 \subseteq I$, $J_1 \subseteq I$ or $J_2 \subseteq I$.

Theorem Let $I_1, \ldots\ldots\ldots, I_n$ be graded ideals of G-graded ring R. Let P be a graded prime ideal such that $\bigcap_{i=1}^{n} I_i \subseteq P$.

Then $I_i \subseteq P$ for some $1 \le i \le n$.

Definition

Let R be a G-graded ring. A graded prime ideal P of R is said to satisfy the condition $(*)$, if $\{I_\alpha\}_{\alpha\in\Delta}$ is a family of graded ideals of R, then P contains $\bigcap_{\alpha\in\Delta} I_\alpha$ only if P contains some I_α. A graded ring R is said to satisfy the condition $(*)$ if all graded prime ideals of R satisfies the condition $(*)$.

Theorem Let R be a G-graded integral domain. If R satisfies the condition $(*)$, then R is a graded field.

Let R and R' be two G-graded rings. A homomorphism of graded rings $\varphi : R \to R'$ is a homomorphism of rings verifying $\varphi(R_g) \subseteq R'_g$ for every $g \in G$.

Theorem

Let R and R' be two G-graded rings and $\varphi : R \to R'$ be an epimorphism of graded rings. Let P' be a graded prime ideal of R'. Then P' is a graded prime ideal of R' if and only if $\varphi^{-1}(P')$ is a graded prime ideal of R.

Theorem Let R and R' be two G-graded rings and $\varphi : R \to R'$ be an epimorphism of graded rings. If R satisfies the condition $(*)$, then R' satisfies the condition $(*)$.

Corollary Let R be a G-graded ring satisfying the condition $(*)$ and I a graded ideal of R. Then R/I satisfies the condition $(*)$.

A G-graded ring R is said to be a graded Artinian (*gr-Artinian*) if satisfies the descending chain condition for graded ideals.

Theorem Let R be a G-graded ring. If R is a graded Artinian ring, then R satisfies the condition $(*)$.

Definition A G-graded ring R is said to satisfy the condition

(#) if P is a graded prime ideal of R and $\{I_\alpha\}_{\alpha\in\Delta}$ is a family of graded ideals of R such that $I_\alpha + P = R$ for all $\alpha \in \Delta$, then $\bigcap_{\alpha\in\Delta} I_\alpha \not\subseteq P$.

Theorem

Let R be a G-graded ring. If R satisfies the condition $(*)$, then R satisfies the condition (#).

Theorem Let R and R' be two G-graded ring and $\varphi : R \to R'$ be an epimorphism of graded ring. If R satisfies the condition (#), then R' satisfies the condition (#).

Theorem Let R be a G-graded ring, and $\{I_\alpha\}_{\alpha\in\Lambda}$ be a family of graded ideals of R. Then the following are equivalent:

i) R satisfies the condition (#).

ii) Every graded maximal ideal M of R satisfies the condition $(*)$.

iii) For any graded maximal ideal M of R, $M + I_\alpha = R$ implies $M + (\bigcap_{\alpha\in\Lambda} I_\alpha) = R$

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