Local loop lemma

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Loop lemmata

A loop lemma: statement of the form

Let $R \subseteq A^2$ (digraph), subdirect, connected, and

- some finiteness assumption
- some structural assumption on R
- R is "compatible" with some "nice" operations

Then R has a loop (ie. $(a, a) \in R$)

Structural assumptions

- R symmetric, contains a triangle
- R symmetric, contains an odd cycle
- R strongly connected, GCD of cycle lengths is 1
- R has algebraic length 1 ($=$ no homomorphism to a cycle)

Algebraic assumptions

R compatible with "nice" operations

- R is compatible with an NU operation
  - R is compatible with $f : A^n \to A$ if $f(R, \ldots, R) \subseteq R$
  - f is NU if $f(x, x, \ldots, x, y, x, x, \ldots, x) \approx x$
- R is compatible with operations satisfying nontrivial idempotent identities
  - nontrivial: not satisfiable by projections
  - idempotent: $f(x, x, \ldots, x) \approx x$
- R is compatible with operations satisfying nontrivial linear identities

R "strongly" compatible with some operations

- R absorbs A^2, ie. $f(R, R, \ldots, R, A^2, R, \ldots, R) \subseteq R$
- R absorbs Δ, ie. $f(R, R, \ldots, R, R \cup \Delta, R, \ldots, R) \subseteq R$

What are loop lemmata good for

- in CS: hardness results (for eg. CSP)
- in UA: Malcev conditions
- Used both directly and as an auxiliary lemma

A loop lemma

Theorem (Hell, Nešetřil'90; Bulatov'05)

Let $R \subseteq A^2$, subdirect, connected, and

- *A finite*
- *R symmetric, contains an odd cycle*
- *R compatible with operations satisfying nontrivial linear identities*

Then R has a loop

Direct applications:

- Dichotomy theorem for CSPs over undirected graphs
- 6-ary Siggers term [Siggers'10]

Siggers from Hell–Nešetřil

Theorem (Siggers'10)

Let **B** *be finite algebra with term operations satisfying nontrivial linear identities.*

Then **B** *has a term operation such that* $s(x,y,x,z,y,z) \approx s(y,x,z,x,z,y)$

Proof.

- $\mathbf{A} :=$ free algebra for $\mathbf{B}$ over $\{x,y,z\}$
- $R \leq \mathbf{A}^2$ generated by $(x,y),(y,x),(x,z),(z,x),(y,z),(z,y)$
- That is

$$R = \{(s(x,y,x,z,y,z), s(y,x,z,x,z,y)) : s \text{ a 6-ary term } \}$$

- Loop in R gives the required term

□

"The loop lemma"

Theorem (Barto,Kozik'09 **SSAOS**)

Let $R \subseteq A^2$, subdirect, connected, and

- *A finite*
- *R has algebraic length 1*
- *R compatible with operations satisfying nontrivial linear identities*
  (or R absorbs Δ)

Then R has a loop

Direct applications:

- Dichotomy theorem for CSPs over smooth digraphs
- 4-ary Siggers term [Kearnes,Marković,McKenzie'14]

Indirect: eg. CSP dichotomy theorems, Valeriote conjecture, ...

End of story?

No!

- pseudo-loop lemmata [Barto, Pinsker'16]
- purely infinite loop lemmata
- local loop lemmata

An infinite loop lemma

Theorem (Olšák)

Let $R \subseteq A^2$, subdirect, connected, and

- *R symmetric, contains an odd cycle*
- *R compatible with an NU operation (or R absorbs A^2 via an idempotent operation)*

Then R has a loop

Application: double-loop lemma $\rightarrow$ weakest idempotent equations

Generalizations: ?

A local loop lemma

Theorem (Barto)

Let $R \subseteq A^2$, subdirect, connected, and

- *R symmetric, contains a triangle a, b, c*
- *R compatible with an operation f such that $f(a, a, \ldots, a, d, a, \ldots, a) = a$ for some neighbor d of a*

Then R has a loop

Generalizations: ?

Applications: ?

Organize the mess!

- Many loop lemmata, incomparable
- Various proof techniques: heavily relational, heavily algebraic
- Common generalization?

Even more mess:

- More binary relations
  - Thm: If $R_1, \ldots, R_n$ absorb Δ via f, then they have a common loop [Barto, Kozik, Willard'12]
- Relations of higher arity (eg. double-loop lemma of [Olšák])
- Intersection properties
- CSP instances
- ... **What is The Loop Lemma from the book**?

Thank you!