Several hierarchies of classes of piecewise testable languages

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An Outline

I. Algebraic Theory of Regular Languages

Varieties of regular languages, syntactic monoid, Eilenberg correspondence and its generalizations.

II. Varieties of Automata

Minimal DFA, closure operators, Eilenberg type correspondence.

III. Automata Enriched with an Algebraic Structure

Ordered automata, meet-automata, DL-automata, Eilenberg type correspondence.

IV. Presentations of Languages via Automata

V. Syntactic Structures

I. Algebraic Theory of Regular Languages

Examples

- Goal of the study: effective characterizations of certain natural classes of regular languages.
- Typical result: a language belongs to a given class iff its syntactic monoid belongs to a certain class of monoids.

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Theorem (Schützenberger – 1966)

A regular language L is star-free if and only if its syntactic monoid is aperiodic.

Theorem (Simon — 1972)

A regular language L is piecewise testable if and only if the syntactic monoid of L is $\mathcal{J}$-trivial.

- General framework – Eilenberg correspondence.

Varieties of Languages

Definition

A variety of languages $\mathcal{V}$ associates to every finite alphabet A a class $\mathcal{V}(A)$ of regular languages over A in such a way that

- $\mathcal{V}(A)$ is closed under finite unions, finite intersections and complements (in particular $\emptyset, A^* \in \mathcal{V}(A)$),
- $\mathcal{V}(A)$ is closed under quotients, i.e.
  $L \in \mathcal{V}(A),\ u, v \in A^*$ implies
  $u^{-1}Lv^{-1} = \{ w \in A^* \mid uwv \in L \} \in \mathcal{V}(A)$,
- $\mathcal{V}$ is closed under preimages in morphisms, i.e.
  $f : B^* \to A^*,\ L \in \mathcal{V}(A)$ implies
  $f^{-1}(L) = \{ v \in B^* \mid f(v) \in L \} \in \mathcal{V}(B)$.

Pseudovarieties of Monoids

Definition

A pseudovariety of finite monoids is a class of finite monoids closed under submonoids, morphic images and products of finite families.

- For a regular language $L \subseteq A^*$ we define a relation $\sim_L$ on A^* by the rule $u \sim_L v$ iff u and v have the same contexts in L.

  Formally: $u \sim_L v \iff \{(p,q) \mid puq \in L\} = \{(p,q) \mid pvq \in L\}$.
- $\sim_L$ is the syntactic congruence of L and $A^*/\sim_L = M_L$ is the syntactic monoid of L.

The Eilenberg Correspondence

- For each pseudovariety of monoids **V**, we denote $\alpha(\mathbf{V})$ the variety of regular languages given by

$$(\alpha(\mathbf{V}))(A) = \{L \subseteq A^* \mid M_L \in \mathbf{V}\} .$$

- For each variety of regular languages $\mathcal{L}$, we denote by $\beta(\mathcal{L})$ the pseudovariety of monoids generated by syntactic monoids M_L, where $L \in \mathcal{L}(A)$ for some alphabet A.

Theorem (Eilenberg – 1976)

The mappings α and β are mutually inverse isomorphisms between the lattice of all pseudovarieties of finite monoids and the lattice of all varieties of regular languages.

A Formal Definition of a DFA

Definition

A deterministic finite automaton over the alphabet A is a five-tuple $\mathcal{A} = (Q, A, \cdot, i, F)$, where

- Q is a nonempty set of states,
- $\cdot : Q \times A \to Q$ is a complete transition function, which can be extended to a mapping $\cdot : Q \times A^* \to Q$ by $q \cdot \lambda = q,\ q \cdot (ua) = (q \cdot u) \cdot a$,
- $i \in Q$ is the initial state,
- $F \subseteq Q$ is the set of final states.

The automaton $\mathcal{A}$ accepts a word $u \in A^*$ iff $i \cdot u \in F$. The automaton $\mathcal{A}$ recognizes the language $L_{\mathcal{A}} = \{u \in A^* \mid i \cdot u \in F\}$.

A Relationship between DFAs and Monoids

If we have a DFA $\mathcal{A} = (Q, A, \cdot, i, F)$, then:

- Each word $u \in A^*$ performs the transformation $\tau_u : Q \to Q$ where $\tau_u(q) = q \cdot u$ for each $q \in Q$.
- The transition monoid of $\mathcal{A}$ is $(\{\tau_u \,|\, u \in A^*\}, \circ)$.
- The transition monoid of the minimal automaton of L is isomorphic to the syntactic monoid M_L of L.

Motivations for a Notion of a Variety of Automata

- Why monoids instead of automata?
  - An equational description of pseudovarieties of monoids by pseudoidentities.
  - Other algebraic constructions, e.g. products (semidirect, wreath, Mal'cev).

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So, basically there are three worlds: classes of languages, classes of (enriched) semiautomata (no initial and no final states) and those of appropriate algebraic structures.

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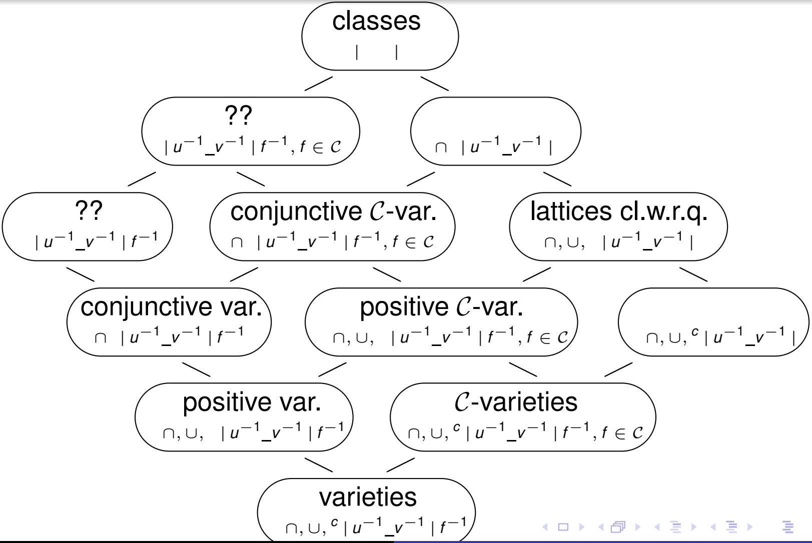
- Polák (1999): Conjunctive (and disjunctive) varieties.
- Straubing (2002): $\mathcal{C}$-varieties of languages.
- Ésik, Larsen (2003): literal varieties of languages.
- Gehrke, Grigorieff, Pin (2008): Lattices of regular languages.

Variants of Varieties of Regular Languages



II. Varieties of Automata

The Construction of a Minimal DFA by Brzozowski

- For a language $L \subseteq A^*$ and $u \in A^*$, we define a left quotient $u^{-1}L = \{ w \in A^* \mid uw \in L \}$.

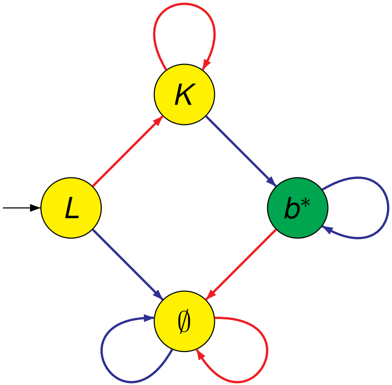
Definition

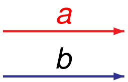
The canonical deterministic automaton of L is $\mathcal{D}_L = (D_L, A, \cdot, L, F)$, where

- $D_L = \{ u^{-1}L \mid u \in A^* \}$,
- $q \cdot a = a^{-1}q$, for each $q \in D_L$, $a \in A$,
- $q \in F$ iff $\lambda \in q$.

- Each state $q = u^{-1}L$ is formed by all words transforming the state q into a final state.

An Example of a Canonical Automaton





$L = a^+ b^+$

$K = a^{-1} L = a^* b^+$

$b^{-1} K = b^*$

Preimages in Morphisms, Varieties of Automata

- Let $f : B^* \to A^*$ be a morphism, We say that $(P, B, \circ)$ is an f-subautomaton of $(Q, A, \cdot)$ if $P \subseteq Q$ and $q \circ b = q \cdot f(b)$ for every $q \in P$, $b \in B$.

Definition

A variety of semiautomata $\mathbb{V}$ associates to every finite alphabet A a class $\mathbb{V}(A)$ of semiautomata (no initial nor final states) over alphabet A in such a way that

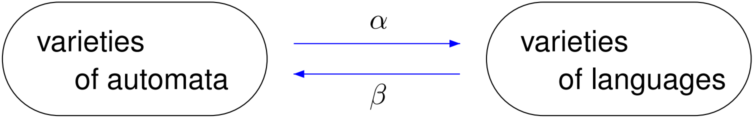
- $\mathbb{V}(A) \neq \emptyset$ is closed under disjoint unions, finite direct products and morphic images,
- $\mathbb{V}$ is closed under f-subautomata.

An Eilenberg Type Correspondence

- For each variety of automata $\mathbb{V}$ we denote by $\alpha(\mathbb{V})$ the variety of regular languages given by

$$(\alpha(\mathbb{V}))(A) = \{L \subseteq A^* \mid \exists \mathcal{A} = (Q, A, \cdot, i, F) : \\ L = L_{\mathcal{A}} \wedge (Q, A, \cdot) \in \mathbb{V}(A)\} .$$

- For each variety of regular languages $\mathcal{L}$ we denote by $\beta(\mathcal{L})$ the variety of automata generated by all DFAs $\mathcal{D}_L$, where $L \in \mathcal{L}(A)$ for some alphabet A.



Theorem (Ésik and Ito, Chaubard, Pin and Straubing)

The mappings α and β are mutually inverse isomorphisms between the lattice of all varieties of automata and the lattice of all varieties of regular languages.

- A version for $\mathcal{C}$-varieties is obvious: we consider f-subautomata (etc.) just for $f \in \mathcal{C}$.
- Ésik and Ito were working with literal varieties (morphisms map letters to letters, i.e $f(B) \subseteq A$) and used disjoint union.
- Chaubard, Pin and Straubing called the automata $\mathcal{C}$-actions and used trivial automata.

An Examples – Acyclic Automata

- One of the conditions in Simon's characterization of piecewise testable languages is that a minimal DFA is acyclic.
- A content $\mathrm{c}(u)$ of a word $u \in A^*$ is the set of all letters occurring in u.
- We say that $(Q, A, \cdot)$ is a acyclic if for each $u \in A^*$ and $q \in Q$ we have

$$q \cdot u = q \implies (\forall a \in \mathrm{c}(u) : q \cdot a = q) .$$

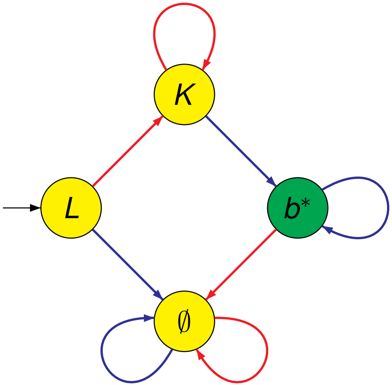
- The class of all acyclic automata is a variety.
- The corresponding variety of languages (well-known): (disjoint) unions of the languages of the form

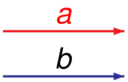
$$A_0^* a_1 A_1^* a_2 A_2^* \dots A_{n-1}^* a_n A_n^*, \quad \text{where} \quad a_i \notin A_{i-1} \subseteq A .$$

An Example – Piecewise Testable Languages

- In DLT'13 we gave an alternative condition for automata recognizing piecewise testable languages.
- We call an acyclic automaton $(Q, A, \cdot)$ locally confluent, if for each state $q \in Q$ and every pair of letters $a, b \in A$, there is a word $w \in \{a, b\}^*$ such that $(q \cdot a) \cdot w = (q \cdot b) \cdot w$.
- A stronger condition: an acyclic automaton $(Q, A, \cdot)$ is confluent, if for each state $q \in Q$ and every pair of words $u, v, \in \{a, b\}^*$, there is a word $w \in \{a, b\}^*$ such that $(q \cdot u) \cdot w = (q \cdot v) \cdot w$.
- Each acyclic automaton is confluent iff it is locally confluent.
- The class of all acyclic confluent automata is a variety which corresponds to the variety of piecewise testable languages.

An Example of a Piecewise Testable Language





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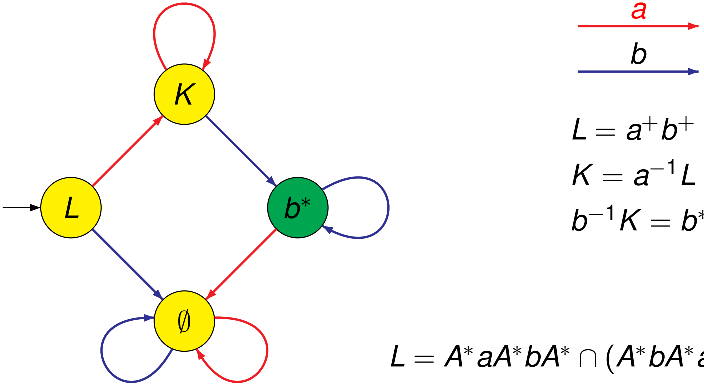
An Example of a Piecewise Testable Language



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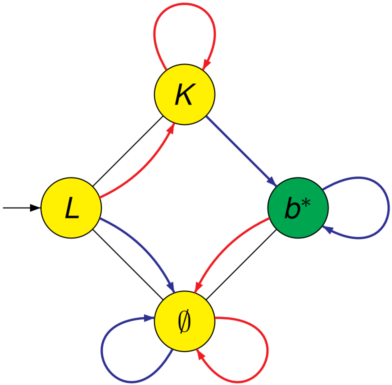
$$L = A^* a A^* b A^* \cap (A^* b A^* a A^*)^c$$

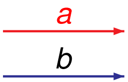
III. Automata Enriched with an Algebraic Structure

A Natural Ordering of the Canonical Automaton

- For a language $L \subseteq A^*$, we have defined a the canonical deterministic automaton: $\mathcal{D}_L = (D_L, A, \cdot, L, F)$, where
  - $D_L = \{\, u^{-1}L \mid u \in A^* \,\}$,
  - $q \cdot a = a^{-1}q$, for each $q \in D_L$, $a \in A$,
  - $q \in F$ iff $\lambda \in q$.
- Therefore states are ordered by inclusion, which means that each minimal automaton is implicitly equipped with a partial order.
- The action by each letter a is an isotone mapping: for all states p, q such that $p \subseteq q$ we have $p \cdot a = a^{-1}p \subseteq a^{-1}q = q \cdot a$.
- The final states form an upward closed subset w.r.t. $\subseteq$.

An Example of an Ordered Automaton





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$L \subseteq K$

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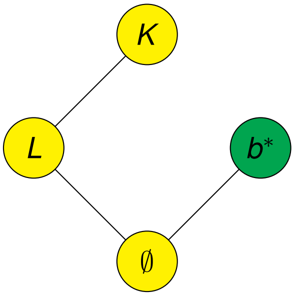


$$L = a^+ b^+$$
$$K = a^{-1} L = a^* b^+$$
$$L \subseteq K$$

An Ordered Automaton

Definition

An ordered automaton over the alphabet A is a six-tuple $\mathcal{A} = (Q, A, \cdot, \leq, i, F)$, where

- $\mathcal{A} = (Q, A, \cdot, i, F)$ is a usual DFA;
- $\leq$ is a partial order;
- an action by every letter is an isotone mapping from the partial ordered set $(Q, \leq)$ to itself;
- F is an upward closed set, i.e. $p \leq q, p \in F \implies q \in F$.

A Transition Monoid of an Ordered Automaton

If we have an ordered automaton $(Q, A, \cdot, \leq)$, then:

- we have defined $\tau_u : Q \rightarrow Q$ transformation by a word $u \in A^*$.
- These transformations can be ordered:

$$\tau_u \leq \tau_v \iff \forall p \in Q : p \cdot \tau_u \leq p \cdot \tau_v .$$

- An ordered transition monoid.
- In particular, the ordered transition monoid of the canonical ordered automaton of L is isomorphic to the syntactic ordered monoid M_L of L.

Algebraic Constructions on Ordered Automata

Definition

A variety of ordered semiautomata $\mathbb{V}$ associates to every finite alphabet A a class $\mathbb{V}(A)$ of ordered semiautomata over alphabet A in such a way that

- $\mathbb{V}(A) \neq \emptyset$ is closed under disjoint union, finite direct products and morphic images,
- $\mathbb{V}$ is closed under f-subautomata.

An Eilenberg Type Correspondence

Theorem (Pin)

There are mutually inverse isomorphisms between the lattice of all varieties of ordered automata and the lattice of all positive varieties of regular languages.

The Level $1/2$

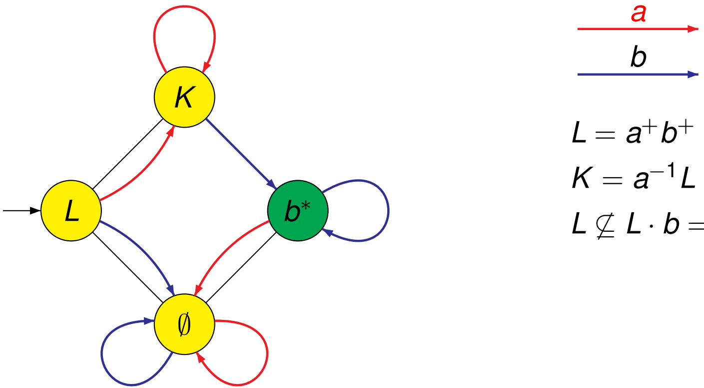
- Piecewise testable languages are Boolean combinations of languages of the form

$$A^*a_1A^*a_2A^* \dots A^*a_\ell A^*, \text{ where } a_1, \dots, a_\ell \in A, \ \ell \geq 0 \, .$$

- Piecewise testable languages form level 1 in Straubing-Thérien hierarchy.
- Level $1/2$ is formed just by finite unions of intersections of languages above.
- The corresponding variety of ordered automata is the class of all ordered automata where actions by letters are increasing mappings. I.e. ordered automata satisfying:

$$\forall q \in Q, a \in A : q \cdot a \geq q \, .$$

An Example of an Ordered Automaton outside 1/2



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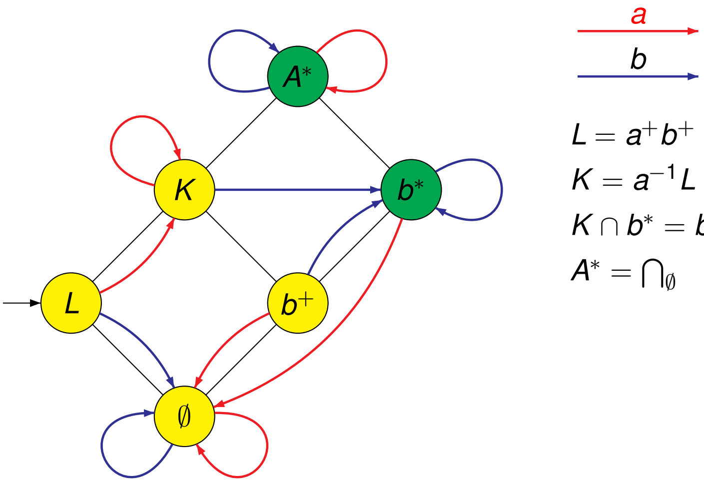
$L \not\subseteq L \cdot b = \emptyset$

III.2 Meet Automata

Intersections of Left Quotients

- For a language $L \subseteq A^*$ we extend the canonical automaton $(D_L, A, \cdot)$, where states are subsets of A^*.
- We can consider intersections of states: $U_L = \{\bigcap_{j\in I} K_j \mid I \text{ finite set }, K_j \in D_L\}$. If $I = \emptyset$ then we put $\bigcap_{j\in I} K_j = A^*$.
- The finite set U_L is equipped with the operation intersection $\cap$ and we can define $(\bigcap_{j\in I} K_j) \cdot a = \bigcap_{j\in I}(K_j \cdot a)$.
- We have the automaton $(U_L, A, \cdot)$ with semilattice operation $\cap$. Moreover, A^* is the largest element in the semilattice $(U_L, \cap)$ and it is an absorbing state in $(U_L, A, \cdot)$.
- Naturally $F = \{K \mid \lambda \in K\}$ is closed w.r.t. $\cap$ and F is upward closed, i.e. F is a filter.

An Example of a Meet Automaton



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$A^* = \bigcap_\emptyset$

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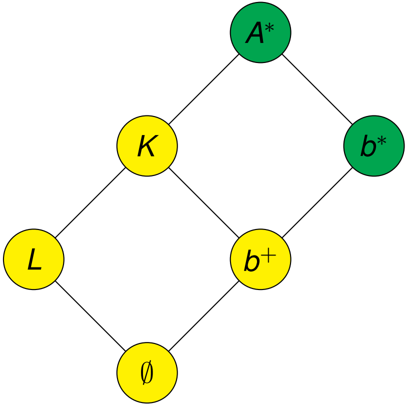

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$$K \cap b^* = b^+$$
$$A^* = \bigcap_\emptyset$$

Meet Automata

Definition

A structure $(Q, A, \cdot, \wedge, \top)$ is a meet semiautomaton if

- $(Q, A, \cdot)$ is a DFA,
- $(Q, \wedge)$ is a semilattice with the largest element $\top$,
- actions by letters are endomorphisms of the semilattice $(Q, \wedge)$, i.e. $\forall p, q \in Q, a \in A : (p \wedge q) \cdot a = p \cdot a \wedge q \cdot a$
- $\top$ is an absorbing state.

This meet semiautomaton recognizes a language L if there are $i, f \in Q$ such that $L = \{u \in A^* \mid i \cdot u \wedge f = f\}$.

Varieties of Meet Automata

Definition

A variety of meet semiautomata $\mathbb{V}$ associates to every finite alphabet A a class $\mathbb{V}(A)$ of meet semiautomata over alphabet A in such a way that

- $\mathbb{V}(A) \neq \emptyset$ is closed under direct finite products and morphic images,
- $\mathbb{V}$ is closed under f-subautomata.

An Eilenberg Type Correspondence

Theorem (Klíma, Polák)

There are mutually inverse isomorphisms between the lattice of all varieties of meet semiautomata and the lattice of all conjunctive varieties of regular languages.

Varieties of Meet Automata – An Example

Example

For each alphabet A, a meet automata $(Q, A, \cdot, \wedge, \top)$ belongs to $\mathbb{S}(A)$ if $\forall q \in Q, a \in A \; : \; q \cdot a = q \cdot a \wedge q$ and

$$\forall q \in Q, a, b \in A \; : \; q \cdot ab = q \cdot a \wedge q \cdot b \, . \quad (*)$$

Then $\mathbb{S}$ is a variety of meet automata and the corresponding conjunctive variety of languages $\mathcal{S}$ satisfies $\mathcal{S}(A) = \{B^* \mid B \subseteq A\} \cup \{\emptyset\}$.

- $\mathcal{S}$ is a conjunctive variety of languages.
- For all $B \subseteq A$, the canonical meet automaton of $L = B^*$ is $\mathcal{U}_L = (U_L, A, \cdot, \cap, A^*)$ and it has just three states: $B^*, A^*, \emptyset$. We see $B^* \cdot a \in \{B^*, \emptyset\}$ and $\mathcal{U}_L$ satisfies $(*)$.

An Example

- Let $(Q, A, \cdot, \wedge, \top)$ be a meet automaton satisfying

$$\forall q \in Q, a, b \in A \cup \{\lambda\} \ : \ q \cdot ab = q \cdot a \wedge q \cdot b \,. \quad (*)$$

  And we choose $i, f \in Q$.
- For $a, b, c \in A$ we have

$$(q \cdot a) \cdot bc = (q \cdot a) \cdot b \wedge (q \cdot a) \cdot c = q \cdot a \wedge q \cdot b \wedge q \cdot c \,.$$

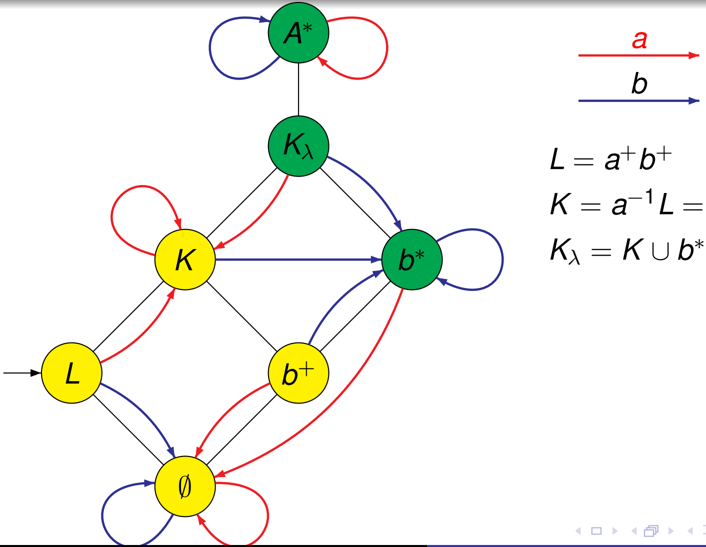
- In general $q \cdot a_1 \dots a_n = q \cdot a_1 \wedge \dots \wedge q \cdot a_n$ and (for $q = i$) we have $a_1 \dots a_n \in L \iff a_1 \in L, \dots, a_n \in L$.
- We can denote $B = L \cap A$ and we have $L = B^*$.
- Note that an equational characterization is that syntactic semiring satisfies the equality $xy = x \wedge y$.

III.3 Lattice automata

The Canonical Lattice Automaton of a Language

- For a language $L \subseteq A^*$ we extend the canonical meet automaton $(U_L, A, \cdot, \wedge, A^*)$ by unions of states: $W_L = \{\bigcup_{j \in I} M_j \mid I \text{ finite set }, M_j \in U_L\}$. If $I = \emptyset$ then we put $\bigcup_{j \in I} M_j = \emptyset$.
- The finite set W_L is equipped with the operations intersection $\cap$ and union $\cup$ (due to distributive laws). We can define $(\bigcup_{j \in I} M_j) \cdot a = \bigcup_{j \in I}(M_j \cdot a)$.
- We have the automaton $(W_L, A, \cdot)$ and a distributive lattice $(W_L, \cap, \cup)$. Moreover, A^* is the largest element, $\emptyset$ is the smallest element – both are absorbing states in $(W_L, A, \cdot)$.
- Naturally $F = \{M \mid \lambda \in M\}$ is closed w.r.t. $\cap$, upward closed, and $M_1 \cup M_2 \in F$ implies $M_1 \in F$ or $M_2 \in F$. I.e. F is an ultrafilter. In other words the intersection of all elements in F (the minimum in F) is join-irreducible.

An Example of a Canonical Lattice Automaton



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$$K_\lambda = K \cup b^* = K + \lambda$$

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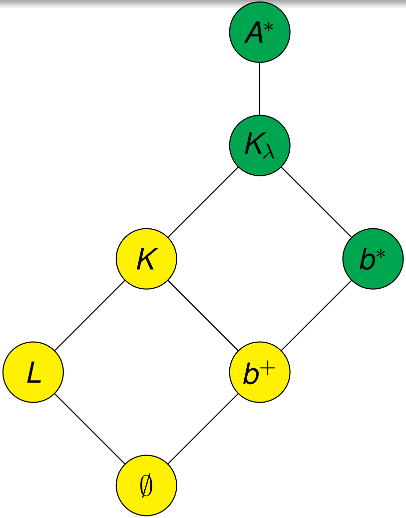

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A Lattice Automata – a Formal Definition

Definition (new)

A structure $(i, P, Q, A, \cdot, \wedge, \vee, \perp, \top)$ is a lattice automaton if

- $i \in P \subseteq Q$,
- $(Q, A, \cdot)$ is a DFA,
- $(Q, \wedge, \vee)$ is a distributive lattice with the minimum element $\perp$ and the largest element $\top$,
- actions by letters are endomorphisms of the lattice $(Q, \wedge, \vee)$,
- $\top$ and $\perp$ are absorbing states,
- P is the set of all states reachable from i,
- the lattice Q is generated by the set P.

Languages Recognized by a Lattice Automaton

- A *DL*-automaton $(i, P, Q, A, \cdot, \wedge, \vee, \perp, \top)$ recognizes a language L if there are $j \in P, f \in Q$ such that f is a join-irreducible and $L = \{u \in A^* \mid j \cdot u \geq f\}$.

An Eilenberg Type Correspondence

Definition (new)

Let $\mathcal{C}$ be a “Straubing” class of morphisms. A weak $\mathcal{C}$-variety of languages $\mathcal{V}$ associates to every finite alphabet A a class $\mathcal{V}(A)$ of regular languages over A in such a way that

- $\mathcal{V}(A)$ is closed under quotients,
- $\mathcal{V}$ is closed under preimages in morphisms from $\mathcal{C}$.

Theorem (new)

There are mutually inverse isomorphisms between the lattice of all $\mathcal{C}$-varieties of lattice semiautomata and the lattice of all weak $\mathcal{C}$-varieties of regular languages.

An Eilenberg Type Correspondence – An Example

Example

Let $\mathcal{V}$ be a class of languages such that
$\mathcal{V}(A) = \{A^*aA^* \mid a \in A\} \cup \{A^*\}$.

- $\mathcal{V}(A)$ is not closed under intersections nor unions, i.e. $\mathcal{V}$ is not a conjunctive (nor disjunctive) variety of languages.
- Let $f : B^* \to A^*$, $a \in A$, $L = A^*aA^*$, then $f^{-1}(L) = B^*DB^*$ where $D = \{d \in B \mid f(d)$ contains $a\}$.
Therefore we should consider only f's such that

$$\forall b, c \in B : b \neq c \implies \mathrm{c}(f(b)) \cap \mathrm{c}(f(c)) = \emptyset\,.$$

- $\mathcal{V}$ is a weak $\mathcal{C}$-variety for such morphisms.

An Example

- Since $(A^*aA^*)^c = B^*$ for $B = A \setminus \{a\}$ we can take the dual condition characterizing the conjunctive variety $\mathcal{S}$ given by $\mathcal{S}(A) = \{B^* \mid B \subseteq A\} \cup \{\emptyset\}$, namely

$$\forall q \in Q, a, b \in A \cup \{\lambda\} \ : \ q \cdot ab = q \cdot a \vee q \cdot b \,. \quad (*)$$

- In particular $q \cdot a \geq q$.
- Another property is the following

$$\forall q \in Q, a, b \in A : a \neq b \implies q \cdot a \wedge q \cdot b = q \,. \quad (**)$$

- If $L = A^*a_1A^* \cup \cdots \cup A^*a_nA^*$ for $a_1, \ldots, a_n$ different letters, $n \geq 2$, then $L \cdot a_1 \cap L \cdot a_2 = A^* \neq L$, i.e. the canonical DL-automaton of L does not satisfy the condition $(**)$.
- The equational description of the property $(**)$ is $x \wedge y = 1$ with substitutions from the restricted class $\mathcal{C}$.

In DLT 2008 we studied the following classes of languages: Let k a natural number, a monomial over an alphabet A is the language $A^*a_1A^*a_2\dots a_lA^*, l \leq k$. Let B_k be the set of all Boolean combinations of monomials, let P_k be finite intersections of finite unions of them and let D_k consist of their finite unions.

We obtained sequences of Boolean varieties, positive varieties and of disjunctive ones. The major problems concern the decidability of a given L in a given class.

Another hierarchies are given by

$$V_k(A) = \{\, A^*B_1A^*B_2\dots B_lA^* \mid l \leq k,\ B_1,\dots,B_l \subseteq A \,\}$$

and $W_k(A) = \{\, P^rQ^s \mid r+s \leq k,\ P,Q \subseteq A \,\}$. The decidability question are immediately solved when looking at minimal DFAs.

IV. Presentations of classes of languages via automata

Using some injectivity conditions

A language L over A is reversible if it is accepted by a NFA $\mathcal{A} = (Q, A, E, I, F)$, $E \subseteq Q \times A \times Q$, not necessarily complete, nor necessarily $| I |= 1$ such that the action of each $a \in A$ on Q is both deterministic a codeterministic.

L is bideterministic if moreover $\mathcal{A}$ can be taken with $| I |=| F |= 1$.

We denote the above classes by $\mathcal{R}$ and $\mathcal{BD}$.

Further, $L \in \mathcal{P}(\mathcal{A})$ if there exists a complete DFA $\mathcal{A}$ recognising L, not necessarily minimal for L, such that there is maximally one absorbing state in $\mathcal{A}$ and this case it is non-final and each $a \in A$ transforms the non-absorbing states injectively.

Using forbidden patterns

Following Ivan, a pattern is a triple $\mathcal{P} = (V, E, l)$ where (V, E) is a finite oriented graph and l labels the edges by variables from the set X. A semiautomaton $\mathcal{A} = (Q, A, \cdot)$ admits $\mathcal{P}$ if there exists an injective mapping $f : V \to Q$ and a mapping $h : X \to A^+$ such that: for each $(p, q) \in E$, we have $f(p) \cdot h(l(p, q)) = f(q)$. Otherwise $\mathcal{A}$ avoids $\mathcal{P}$.

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Examples:
$\mathcal{P}_f = (\{p, q\}, \{(p, p), (q, q)\}, l),\ l(p, p) = x, l(q, q) = y.$
$\mathcal{P}_d = (\{p, q\}, \{(p, p), (q, q)\}, l),\ l(p, p) = l(q, q) = x.$
$\mathcal{P}_r = (\{p, q\}, \{(p, p), (p, q)\}, l),\ l(p, p) = x, l(p, q) = y.$
$\mathcal{P}_g = (\{p, q\}, \{(p, p), (p, q), (q, q)\}, l),\ l(p, p) = x, l(p, q) = y, l(q, q) = x.$

Using forbidden patterns II

Ivan shows that the languages for which the minimal complete DFAs avoids those patterns are exactly finite or cofinite, definite, reverse definite and generalized definite languages, respectively.

In several Pin's papers one can find another kind of conditions: In a DFA $\mathcal{A} = (Q, A, \cdot, i, F)$ there are

- (1) no $p, q, r \in Q$, $p \neq q \neq r$ for which there are $u, v \in A^*$ with $p \cdot u = q \cdot u = q, q \cdot v = r$,
- (2) no $p, q, r, s, t \in Q$, $p \neq t$ for which there are $u, v \in A^*$ such that
  $p{\cdot}u = q{\cdot}u = p, q{\cdot}v = r{\cdot}v = q, r{\cdot}u = s{\cdot}u = s, s{\cdot}v = t{\cdot}v = t,$
- (3) no $p, q, r, s \in Q, p \notin F, s \in F$ for which there are $u, v \in A^*$ such that $q \cdot v = p, q \cdot u = r \cdot u = r, r \cdot v = s,$

Using forbidden patterns III

- (4) no $p, q, r \in Q$, $q \neq r$ for which there are $u, v \in A^*$ with $p \cdot v = p, p \cdot u = q \cdot u = q, q \cdot v = r \cdot v = r$.

Pin also observed that (2) and (3) in minimal DFA charactrerize the class $\mathcal{P}$ from the last subsection.

Using meet automata

See OK and LP, On varieties of meet automata, Theor. Computer Science (407), 2008

V. Syntactic Structures

Basic algebras

Monoids. Let F be the free groupoid over an alphabet A with neutral element λ. We define inductively the actions of elements of F on 2^{A^*} :

$$L \circ \lambda = L, \; L \circ a = a^{-1}L, \; L \circ (u \cdot v) = (L \circ u) \circ v\,.$$

Put $u \,\rho\, v$ iff (for each $L \subseteq A^*$, we have $L \circ u = L \circ v$). Clearly, $A^* = F/\rho$ is a free monoid over A.

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Semirings. Let F' be the absolutely free over A with respect to the operational symbols $\cdot, \lambda$ and $\wedge$.

$$\ldots,\ L \circ (u \wedge v) = (L \circ u) \cap (L \circ v)\,.$$

Put $u\ \rho'\ v$ iff (for each $L \subseteq A^*$, we have $L \circ u = L \circ v$). Clearly, $A^\square = F'/\rho'$ is a free semiring over A.
Its elements can be represented as finite subsets of A^* with the obvious multiplication and the operation of the union.

lattice-algebras. Let F'' be the absolutely free over A with respect to the operational symbols $\cdot, \lambda, \wedge$ and $\vee$.

$$..., \; L \circ (u \vee v) = (L \circ u) \cup (L \circ v)\,.$$

Put $u \;\rho''\; v$ iff (for each $L \subseteq A^*$, we have $L \circ u = L \circ v$). On can show that $A^\diamond = F''/\rho''$ is a free distributive lattice over A^*. The elements can be represented as

$$\{\{u_{1,1}, \ldots, u_{1,\ell_1}\}, \ldots, \{u_{k,1}, \ldots, u_{k,\ell_k}\}\}, \; u_{i,j} \in A^*$$

with incomparable inner sets. The interpretation is $(u_{1,1} \wedge \ldots) \vee \cdots \vee (u_{k,1} \wedge \ldots)$. It is equiped also with a (mysterious) multiplication, namely extend the multiplication from A^* to $A^\diamond$ using

$$U \cdot (V \wedge W) = U \cdot V \wedge U \cdot W, \; (U \wedge V) \cdot w = U \cdot w \wedge V \cdot w$$

for $U, V, W \in A^\diamond$, $w \in A^*$. Similarly for the operation $\vee$.

Syntactic structures

Let $L \subseteq A^*$ be a regular language.

Monoids. For $u, v \in A^*$, put

$$u \sim v \text{ iff } \forall\, p, q \in A^* (\, puq \in L \Longleftrightarrow pvq \in L \,)\,.$$

Equivalently, $\forall\, p \in A^*\ p^{-1}L \circ u = p^{-1}L \circ v$. Then $A^*/\sim$ is the syntactic monoid of L.

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Semirings. For $U = \{u_1, \dots, u_k\},\ V = \{v_1, \dots, v_\ell\} \in A^\Box$, put $U \sim' V$ iff

$$\forall\, p, q \in A^* (\, pu_1q \in L, \dots, pu_kq \in L \iff pv_1q \in L, \dots, pv_\ell q \in L\,)\,.$$

Equivalently, $\forall\, p \in A^*\ p^{-1}L \circ U = p^{-1}L \circ V$. Then $A^\Box/\sim'$ is the syntactic semiring of L.

lattice-algebras.

Let

$$\mathcal{U} = \{U_1, \ldots, U_k\},$$

$$U_1 = \{u_{1,1}, \ldots, u_{1,m_1}\}, \ldots, U_k = \{u_{k,1}, \ldots, u_{k,m_k}\},$$

$$\mathcal{V} = \{V_1, \ldots, V_\ell\},$$

$$V_1 = \{v_{1,1}, \ldots, v_{1,n_1}\}, \ldots, V_\ell = \{v_{\ell,1}, \ldots, v_{\ell,n_\ell}\}.$$

Put $\mathcal{U} \sim'' \mathcal{V}$ iff $\forall\, p, q \in A^*$

$(\, pU_1q \in L$ or $\ldots$ or $pU_kq \in L \iff pV_1q \in L$ or $\ldots$ or $pV_\ell q \in L\,)$.

Equivalently, $\forall\, p \in A^*\ p^{-1}L \circ \mathcal{U} = p^{-1}L \circ \mathcal{V}$. Then $A^\diamond / \sim''$ is the syntactic lattice-algebra of L.

General lattice-algebras

Not every finite monoid is isomorphic to a syntactic one,...
So, a finite **lattice-algebra** is a finite distributive lattice with a multiplication and and a chosen subset $(L, \cdot, \wedge, \vee, P)$ such that the lattice $(L, \wedge, \vee)$ is generated by P^* and the distibributivities above hold for L and P^*.
Acceptance ...
$\mathcal{C}$-pseudovarieties ...
Kunc ...
Eilenberg ...