Substructural Logics

A Logical Glimpse at Residuated Lattices

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Outline

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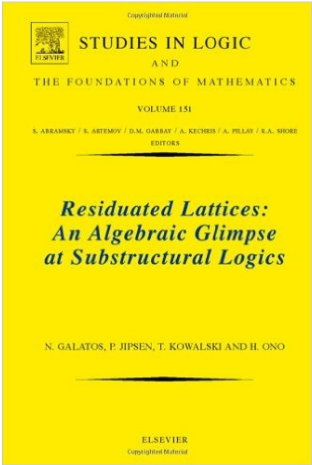

Galatos–Jipsen–Kowalski–Ono.
Residuated Lattices:
An Algebraic Glimpse
at Substructural Logics.
Elsevier, 2007

Non-associative residuated lattices [Galatos–Ono. APAL 2010]

A pointed residuated lattice-ordered groupoid with unit $\boldsymbol{A}$ is algebra of a type $\mathcal{L}_{\mathrm{SL}} = \{\&, \backslash, /, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$:

- $\langle A, \wedge, \vee \rangle$ is a lattice
- $\langle A, \&, \mathbf{1} \rangle$ is a groupoid with unit $\mathbf{1}$
- for each $x, y, z \in A$:

$$x \,\&\, y \leq z \quad \text{IFF} \quad x \leq z \,/\, y \quad \text{IFF} \quad y \leq x \,\backslash\, z$$

For simplicity we will speak about SL-algebras

SL-algebras form a variety, we will denote it as $\mathbb{SL}$.

Notable examples

- FL-algebras = pointed residuated lattices = 'associative' SL-algberas
- Algebras of relations, where $\&$ is relational composition and

$$R \setminus S = (R \,\&\, R^c)^c \qquad S \,/\, R = (S^c \,\&\, R)^c$$

- ℓ-groups, where $a \setminus b = a^{-1} \,\&\, b$ and $b \,/\, a = b \,\&\, a^{-1}$
- Powersets of monoids, where

$$X \setminus Y = \{z \mid X \,\&\, \{z\} \subseteq Y\} \qquad Y \,/\, X = \{z \mid \{z\} \,\&\, X \subseteq Y\}$$

- Ideals of a ring ...

Classes of residuated structures

Any quasivariety of $\mathbb{SL}$-algberas with possible additional operators will be called a class of residuated structures

Classes of residuated structures

Any quasivariety of $\mathbb{SL}$-algberas with possible additional operators will be called a class of residuated structures

- Subvariets of $\mathbb{SL}$, where $\&$ is associative, commutative, idempotent, divisible, etc.
- Integral $\mathbb{SL}$-algebras: those where $\mathbf{1}$ is a top element of A
- Semilinear classes (those generated by their linearly ordered members)
- Hájek's $\mathbb{BL}$-algebras (associative, commutative, integral, divisible, semilinear $\mathbb{SL}$-algebras)
- $\mathbb{MV}$-algebras ($\mathbb{BL}$-algebras where $(x \to \mathbf{0}) \to \mathbf{0} = x$)
- Boolean algebras (idempotent $\mathbb{MV}$-algebras)

Plus any of these with additional operators . . .

Outline

A short dictionary

Logic	Algebra
language $\mathcal{L}$	type
set of formulas $\mathrm{Fm}_{\mathcal{L}}$	set of terms
Lindenbaum algebra $\mathbf{Fm}_{\mathcal{L}}$	term algebra of type $\mathcal{L}$
$\mathcal{L}$-substitution σ	endomorphism $\mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$
$\boldsymbol{A}$-evaluation e	homomorphism $\mathbf{Fm}_{\mathcal{L}} \to \boldsymbol{A}$

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A logic L is an algebraic closure operator C on $\mathrm{Fm}_{\mathcal{L}}$ s.t. for each substitution σ:

$$\sigma[C(T)] \subseteq C(\sigma[T])$$

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Definition

A logic L is a finitary structural consequence relation, i.e., a relation between sets of formulae and formulae s.t.:

- $T, \varphi \vdash_{\mathrm{L}} \varphi$ (Reflexivity)
- If $S \vdash_{\mathrm{L}} \psi$ and $T, \psi \vdash_{\mathrm{L}} \varphi$, then $T, S \vdash_{\mathrm{L}} \varphi$ (Cut)
- If $T \vdash_{\mathrm{L}} \varphi$, then $\sigma[T] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ (Structurality)
- If $T \vdash_{\mathrm{L}} \varphi$, then there is a finite $T' \subseteq T$ such that $T' \vdash_{\mathrm{L}} \varphi$ (Finitarity)

Axiomatization

Axiomatic system $\mathcal{AS}$ if given by a set of

- axioms Ax, i.e. a set formulas closed under arbitrary substitution,
- rules Ru, i.e. a set of pairs $T \rhd \varphi$ for some finite set $T \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ (again closed under arbitrary substitution)

Proof: of a formula φ from a set of formulae T in $\mathcal{AS}$ is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ s.t.

- $\varphi_n = \varphi$ and for each $i \leq n$ either $\varphi_i \in \mathrm{Ax} \cup T$ or
- there is a set $S \subseteq \{\varphi_1, \ldots, \varphi_{i-1}\}$ and a rule $S \rhd \varphi_i \in \mathrm{Ru}$.

Theorem

We write $T \vdash^{\mathcal{AS}} \varphi$ if there is a proof of φ from T in $\mathcal{AS}$.
$\vdash^{\mathcal{AS}}$ is the least logic L such that

- $\emptyset \vdash_{\mathrm{L}} \varphi$ *for each* $\varphi \in \mathrm{Ax}$
- $S \vdash_{\mathrm{L}} \varphi$ *for each* $S \rhd \varphi \in \mathrm{Ru}$

The logic of $\mathbb{SL}$-algebras

Theorem

The relation $\vdash_{\mathrm{SL}}$ *defined as:*

$$T \vdash_{\mathrm{SL}} \varphi \quad \textit{iff} \quad \{\psi \wedge \mathbf{1} \approx \mathbf{1} \mid \psi \in T\} \models_{\mathbb{SL}} \varphi \wedge \mathbf{1} \approx \mathbf{1}$$

is a logic.

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is a logic.

Axiomatization $\mathcal{SL}$ for SL [Galatos–Ono. APAL, 2010]

Axioms:

$$\begin{array}{lll}
\varphi \wedge \psi \setminus \varphi & \varphi \wedge \psi \setminus \psi & (\chi \setminus \varphi) \wedge (\chi \setminus \psi) \setminus (\chi \setminus \varphi \wedge \psi) \\
\varphi \setminus \varphi \vee \psi & \psi \setminus \varphi \vee \psi & (\varphi \setminus \chi) \wedge (\psi \setminus \chi) \setminus (\varphi \vee \psi \setminus \chi) \\
\varphi \setminus ((\psi \,/\, \varphi) \setminus \psi) & \psi \setminus (\varphi \setminus \varphi \,\&\, \psi) & (\chi \,/\, \varphi) \wedge (\chi \,/\, \psi) \setminus (\chi \,/\, \varphi \vee \psi) \\
\mathbf{1} & \mathbf{1} \setminus (\varphi \setminus \varphi) & \varphi \setminus (\mathbf{1} \setminus \varphi)
\end{array}$$

Rules:

$$\begin{array}{ll}
\{\varphi, \varphi \setminus \psi\} \rhd \psi & \{\varphi\} \rhd (\varphi \setminus \psi) \setminus \psi \\
\{\varphi \setminus (\psi \setminus \chi)\} \rhd \psi \setminus (\chi \,/\, \varphi) & \{\psi \,/\, \varphi\} \rhd \varphi \setminus \psi \\
\{\varphi \setminus \psi\} \rhd (\psi \setminus \chi) \setminus (\varphi \setminus \chi) & \{\psi \setminus \chi\} \rhd (\varphi \setminus \psi) \setminus (\varphi \setminus \chi) \\
\{\varphi, \psi\} \rhd \varphi \wedge \psi & \{\psi \setminus (\varphi \setminus \chi)\} \rhd \varphi \,\&\, \psi \setminus \chi
\end{array}$$

A formal definition of substructural logics

We write $\varphi \to \psi$ instead of $\varphi \backslash \psi$

$\varphi \leftrightarrow \psi$ instead of $(\varphi \to \psi) \wedge (\psi \to \varphi)$

Definition

A logic L in a language $\mathcal{L}$ is a substructural logic if

- $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$
- If $T \vdash_{\mathrm{SL}} \varphi$, then $T \vdash_{\mathrm{L}} \varphi$
- for each n, $i < n$, and each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$ holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \leftrightarrow c(\chi_1, \dots \chi_i, \psi, \dots, \chi_n)$$

Note: the last condition can be prove for all connectives of $\mathcal{L}_{\mathrm{SL}}$

From substructural logics to classes of residuated structures

Theorem

Let L *be a substructural logic. An* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *is an* L*-algebra*, $\mathbb{A} \in \mathbb{Q}_{\mathrm{L}}$, *whenever*

1. *its* $\mathcal{L}_{\mathrm{SL}}$*-reduct is an* SL*-algebra and*
2. $T \vdash_{\mathrm{L}} \varphi$ *implies that* $\{\psi \geq \mathbf{1} \mid \psi \in T\} \models_{\boldsymbol{A}} \varphi \geq \mathbf{1}$

Then $\mathbb{Q}_{\mathrm{L}}$ *is a class of residuated structures and*

$$T \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \{\psi \geq \mathbf{1} \mid \psi \in T\} \models_{\mathbb{Q}_{\mathrm{L}}} \varphi \geq \mathbf{1}$$

From substructural logics to classes of residuated structures and back

Theorem

Let $\mathbb{Q}$ be a class of residuated structures of type $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$. Then the relation $\mathrm{L}_{\mathbb{Q}}$ defined as:

$$T \vdash_{\mathrm{L}_{\mathbb{Q}}} \varphi \quad \textit{iff} \quad \{\psi \geq \mathbf{1} \mid \psi \in T\} \models_{\mathbb{Q}} \varphi \geq \mathbf{1}$$

is a substructural logic. And

$$E \models_{\mathbb{Q}} \alpha \approx \beta \quad \textit{iff} \quad \{\varphi \leftrightarrow \psi \mid \varphi \approx \psi \in E\} \vdash_{\mathrm{L}_{\mathbb{Q}}} \alpha \leftrightarrow \beta$$

It gets even better

Theorem

The operators $\mathbb{Q}_\star$ and $\mathrm{L}_\star$ are dual-lattice isomorphisms between the lattice of substructural logics in language $\mathcal{L}$ and the lattice of subquasivarieties of SL-algebras with operators $\mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$.

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$$\varphi \vdash_{\mathrm{L}} \varphi \wedge \mathbf{1} \leftrightarrow \mathbf{1} \qquad \varphi \wedge \mathbf{1} \leftrightarrow \mathbf{1} \vdash_{\mathrm{L}} \varphi$$

$$\varphi \approx \psi \models_{\mathbb{Q}} (\varphi \leftrightarrow \psi) \wedge \mathbf{1} \approx \mathbf{1} \qquad (\varphi \leftrightarrow \psi) \wedge \mathbf{1} \approx \mathbf{1} \models_{\mathbb{Q}} \varphi \approx \psi$$

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Note: all these results are just particularization of know facts of Abstract Algebraic Logic (AAL)

Examples of substructural logics

- Ono's substructural logics including classical and intuitionistic logic
- expansions by additional connectives, e.g. (classical) modalities, exponentials in linear logic and Baaz's Delta in fuzzy logics
- NOT IN THIS TALK: the fragments of the logics above to languages containing implication, such as BCK, BCI, psBCK, BCC, hoop logics, etc.

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Special axioms:

usual name	**s**	axioms
associativity	a	$(\varphi \mathbin{\&} \psi) \mathbin{\&} \chi \leftrightarrow \varphi \mathbin{\&} (\psi \mathbin{\&} \chi)$
exchange	e	$\varphi \mathbin{\&} \psi \to \psi \mathbin{\&} \varphi$
contraction	c	$\varphi \to \varphi \mathbin{\&} \varphi$
weakening	w	$\varphi \mathbin{\&} \psi \to \psi$ and $\mathbf{0} \to \varphi$

Logic given by these axioms; let $X \subseteq \{\mathrm{e}, \mathrm{c}, \mathrm{w}\}$ we define logics

- SL_X axiomatized by adding axioms from X of those of SL
- FL_X axiomatized by adding associativity to SL_X

Outline

Proof by cases

For classical or intuitionistic logic we have:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma \cup \{\varphi \vee \psi\} \vdash_{\mathrm{L}} \chi}$$

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But in $\mathrm{FL_e}$ it would entail $\varphi \vee \psi \vdash_{\mathrm{FL_e}} (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})$, i.e.,

$$(\varphi \vee \psi) \wedge \mathbf{1} \approx \mathbf{1} \models_{\mathbb{Q}_{\mathrm{FL_e}}} (\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1}) \approx \mathbf{1}$$

which can be easily refuted

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On the other hand we can show that:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL_e}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{FL_e}} \chi}{\Gamma \cup \{(\varphi \wedge \mathbf{1}) \vee (\psi \wedge \mathbf{1})\} \vdash_{\mathrm{FL_e}} \chi}$$

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Results in this section are from: Czelakowski. *Protoalgebraic Logic*, 2000 and C–Noguera. The proof by cases property and its variants in structural consequence relations. *Studia Logica*, 2013.

Generalized disjunctions

Let $\nabla(p, q, \vec{r})$ be a set of formulas. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \vec{\alpha}) \mid \vec{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}.$$

Definition

∇ is a p-disjunction if:

(PD) $\varphi \vdash_{\mathrm{L}} \varphi\nabla\psi$ and $\psi \vdash_{\mathrm{L}} \varphi\nabla\psi$

(PCP) $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ and $\Gamma, \psi \vdash_{\mathrm{L}} \chi$ implies $\Gamma, \varphi\nabla\psi \vdash_{\mathrm{L}} \chi$

Definition

A logic L is a p-disjunctional if it has a p-disjunction.

We drop the prefix 'p-' if there are no parameters $\vec{r}$ in ∇

Separating examples

Example

- $\vee$ is a disjunction in $\mathrm{FL_{ew}}$
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- $\vee$ is a disjunction in $\mathrm{FL_{ew}}$
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- No single formula is a disjunction in $\mathrm{G}_{\to}$

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  but the set $\{(p \to q) \to q, (q \to p) \to p\}$ is
- No finite set of formulas is a disjunction in K

Separating examples

Example

- $\vee$ is a disjunction in $\mathrm{FL_{ew}}$
- $\vee$ is not a disjunction in $\mathrm{FL_e}$, but $(p \wedge \mathbf{1}) \vee (q \wedge \mathbf{1})$ is
- No single formula is a disjunction in $\mathrm{G}_{\rightarrow}$ but the set $\{(p \rightarrow q) \rightarrow q, (q \rightarrow p) \rightarrow p\}$ is
- No finite set of formulas is a disjunction in K but the set $\{\Box^n p \vee \Box^m q \mid n, m \geq 0\}$ is
- No set of formulas in two variables is a disjunction in $\mathrm{IPC}_{\rightarrow}$

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Example

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- No set of formulas in two variables is a disjunction in $\mathrm{IPC}_{\rightarrow}$
  but the formula $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow r)$ is a p-disjunction.

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  but the formula $(p \to r) \to ((q \to r) \to r)$ is a p-disjunction.

Conjecture: The logics SL and FL are not disjunctional;
later we show that they are p-disjunctional

A little detour to AAL 1: filters

Definition

Let L be a substructural logic in $\mathcal{L}$ and $\boldsymbol{A}$ be an $\mathcal{L}$-algebra. A set $F \subseteq A$ is called L-filter on $\boldsymbol{A}$ if:

$$T \vdash_{\mathrm{L}} \varphi \text{ implies that for each } \boldsymbol{A}\text{-evaluation } e \text{ if } e[T] \subseteq F \text{ then } e(\varphi) \in F$$

- If the $\mathcal{L}_{\mathrm{SL}}$-reduct of $\boldsymbol{A}$ is an SL-algebra then:

  $$\boldsymbol{A} \text{ is an } \mathrm{L}\text{-algebra IFF the set } [\mathbf{1}\rangle \text{ is an } \mathrm{L}\text{-filter}$$

- If $\boldsymbol{A}$ is an L-algebra, then $[\mathbf{1}\rangle = \{x \in A \mid \mathbf{1} \leq x\}$ is the least L-filter
- Filters on $\boldsymbol{A}$ form an algebraic closure system

  by $\mathrm{Fi}(X)$ we denote the filter generated by X
- Filters on $\mathbf{Fm}_{\mathcal{L}}$ are the closure system corresponding to L
- When seen as a lattice they are isomorphic to the lattice of

  $\mathbb{Q}_{\mathrm{L}}$-relative congruences on $\boldsymbol{A}$

Filters in p-disjunctional logics

Theorem

Let L *be a logic with a p-disjunction* ∇*. Then for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $X, Y \cup \{x, y\} \subseteq A$*:*

$$\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) = \mathrm{Fi}(X, x\nabla^{\boldsymbol{A}} y)$$

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Theorem

Let L *be a substructural logic. TFAE:*

1. L *is p-disjunctional*
2. *The lattice of all* L*-filters on any* $\mathcal{L}$*-algebra is distributive*
3. $\mathbb{Q}_{\mathrm{L}}$ *is relative-congruence-distributive*

Corollary

For each subvariety $\mathbb{V}$ *of* $\mathbb{SL}$, $\mathrm{L}_{\mathbb{V}}$ *is p-disjunctional logic*

A little detour to AAL 2: RFSI algebras

Let us by $\mathbb{Q}_{\mathrm{RFSI}}$ denote that class of $\mathbb{Q}$-relatively finitely subdirectly irreducible (RFSI) L-algebras. We know that:

$$T \vdash_{\mathrm{L}} \varphi \qquad \text{iff} \qquad \{\psi \geq \mathbf{1} \mid \psi \in T\} \models_{(\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}} \varphi \geq \mathbf{1}$$

$\boldsymbol{A} \in (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}$ iff the the filter $[\mathbf{1}\rangle$ is finitely meet irreducible, i.e., there is no pair of filters $F, G \supset [\mathbf{1}\rangle$ s.t. $F \cap G = [\mathbf{1}\rangle$.

∇-prime filters

Definition

A filter F on $\boldsymbol{A}$ is ∇-prime if for every $a, b \in A$, $a \nabla^{\boldsymbol{A}} b \subseteq F$ implies $a \in F$ or $b \in F$.

Theorem

Let ∇ be a p-disjunction in L and $\boldsymbol{A}$ and L-algebra. Then $\boldsymbol{A} \in (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}$ iff the filter $[\mathbf{1}\rangle$ is ∇-prime.

Proof:

Assume that $\boldsymbol{A}$ is not RFSI: there are $F_i \supset [\mathbf{1}\rangle$ s.t. $[\mathbf{1}\rangle = F_1 \cap F_2$. Let $a_i \in F_i \setminus [\mathbf{1}\rangle$. Thus $a_1 \nabla a_2 \subseteq F_i$, i.e., $[\mathbf{1}\rangle$ is not ∇-prime

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Assume that $[\mathbf{1}\rangle$ is not ∇-prime: there are $x, y \not\geq \mathbf{1}$ s.t. $x\nabla y \subseteq [\mathbf{1}\rangle$. Then $\mathrm{Fi}(x), \mathrm{Fi}(y) \supset [1\rangle$ and:

$$[\mathbf{1}\rangle = \mathrm{Fi}(x\nabla y) = \mathrm{Fi}(x) \cap \mathrm{Fi}(y) \qquad \text{i.e., } \boldsymbol{A} \text{ is not RFSI.}$$

A little detour to AAL 3: simple observations

Let $\mathcal{AX}$ be an axiomatic system of a logic L, then F is an L filter iff it is an upset containing $\mathbf{1}$ and for each rule $T \rhd \varphi$ we have:

for each $\boldsymbol{A}$-evaluation e if $e[T] \subseteq F$ then $e(\varphi) \in F$

$\mathrm{L} + \mathcal{A}$ is the extension of L by axioms from $\mathcal{A}$.

$\mathbb{Q}_{\mathrm{L}+\mathcal{A}}$ is a relative subvariety of $\mathbb{Q}_{\mathrm{L}}$ axiomatized by $\{\varphi \geq \mathbf{1} \mid \varphi \in \mathcal{A}\}$

Positive universal formulas

A *positive universal formula* is built from equations using conjunction and disjunction.

Lemma (Galatos. Studia Logica, 2004)

A positive universal formula C *is equivalent the formula* $\bigvee_{\varphi \in F_C} \mathbf{1} \leq \varphi$

Lemma

Let L be a logic, ∇ a p-disjunction, C a positive universal formula, and $\boldsymbol{A}$ an L-algebra.

- If $\boldsymbol{A} \models C$, then $e[\nabla_{\varphi \in F_C} \varphi] \geq \mathbf{1}$ for each $\boldsymbol{A}$-evaluation e.

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Let L be a logic, ∇ a p-disjunction, C a positive universal formula, and $\boldsymbol{A}$ an L-algebra.

- If $\boldsymbol{A} \models C$, then $e[\nabla_{\varphi \in F_C} \varphi] \geq \mathbf{1}$ for each $\boldsymbol{A}$-evaluation e.
- Furthermore, if $[\mathbf{1}\rangle$ a ∇-prime, then the converse holds as well.

Logics given by positive universal classes of algebras

Theorem

Let L *be a logic with a p-disjunction* ∇ *and* $\mathcal{C}$ *a set of positive universal formulas. Then:*

$$\mathrm{L}_{\mathbf{Q}(\{\boldsymbol{A} \text{ an } \mathrm{L}\text{-algebra} \mid \boldsymbol{A} \models \mathcal{C}\})} = \mathrm{L} + \{\nabla_{\varphi \in F_C} \varphi \mid C \in \mathcal{C}\}.$$

Proof

We set $\mathrm{L}' = \mathrm{L} + \{\nabla_{\varphi \in F_C} \varphi \mid C \in \mathcal{C}\}$; $\mathbb{U} = \{\boldsymbol{A} \text{ an } \mathrm{L}\text{-algebra} \mid \boldsymbol{A} \models \mathcal{C}\}$.

Clearly $\mathbb{U} \subseteq \mathbb{Q}_{\mathrm{L}'}$, so $\mathbf{Q}(\mathbb{U}) \subseteq \mathbb{Q}_{\mathrm{L}'}$ and so $\mathrm{L}' \subseteq \mathrm{L}_{\mathbf{Q}(\mathbb{U})}$.

Conversely, assume that $T \nvdash_{\mathrm{L}'} \varphi$. There is an $\boldsymbol{A} \in (\mathbb{Q}_{\mathrm{L}'})_{\mathrm{RFSI}}$ where $[\mathbf{1}\rangle$ a ∇-prime (because L' is axiomatic extension of L and so ∇ is p-disjunction in L') and an $\boldsymbol{A}$-model of T s.t. $e(\varphi) \ngeq \mathbf{1}$.
Then $\boldsymbol{A} \in \mathbb{U}$ and so $T \nvdash_{\mathrm{L}_{\mathbf{Q}(\mathbb{U})}} \varphi$, i.e. $\mathrm{L}_{\mathbf{Q}(\mathbb{U})} \subseteq \mathrm{L}'$.

Quasivarieties given by positive universal classes of algebras

Corollary

Let L be a logic with a p-disjunction ∇. The quasivariety generated by the class of L-algebras satisfying a set of positive universal formulas $\mathcal{C}$ is axiomatized (relative to $\mathbb{Q}_{\mathrm{L}}$) by:

$$\{\varphi \geq \mathbf{1} \mid C \in \mathcal{C} \text{ and } \varphi \in \bigtriangledown_{\psi \in F_C} \psi\}$$

Note that the axiomatized quasivariety is relative subvariety.

Quasivarieties given by positive universal classes of algebras

Corollary

Let L *be a logic with a p-disjunction* ∇. *The quasivariety generated by the class of* L*-algebras satisfying a set of positive universal formulas* $\mathcal{C}$ *is axiomatized (relative to* $\mathbb{Q}_{\mathrm{L}}$*) by:*

$$\{\varphi \geq \mathbf{1} \mid C \in \mathcal{C} \text{ and } \varphi \in \nabla_{\psi \in F_C} \psi\}$$

Note that the axiomatized quasivariety is relative subvariety.

A remark: this result can be generalized to Qvs generated by classes of RFSI L-algebras satisfying a set of disjunctions of quasiequations.

Intersection of relative subvarieties

Corollary

Let L *be a logic with a p-disjunction* ∇. *The join of two relative subvarieties* $\mathbb{Q}_{\mathrm{L}}$ *axiomatized (relative to* $\mathbb{Q}_{\mathrm{L}}$*) by* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *is axiomatized (relative to* $\mathbb{Q}_{\mathrm{L}}$*) by:*

$$\{\chi \geq \mathbf{1} \mid \varphi_1 \approx \psi_1 \in \mathcal{E}_1, \varphi_2 \approx \psi_2 \in \mathcal{E}_2, \text{ and } \chi \in (\varphi_1 \leftrightarrow \psi_1)\nabla(\varphi_2 \leftrightarrow \psi_2)\}$$

Note that it is the join both in the lattice of subquasivarieties and relative subvarieties

Proof

Assume that the set of variables of $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint.
Then $\boldsymbol{A} \in \mathbb{Q}_1 \cup \mathbb{Q}_2$ iff $\boldsymbol{A} \models (\varphi_1 \approx \psi_1) \vee (\varphi_2 \approx \psi_2)$ for each

$$\varphi_1 \approx \psi_1 \in \mathcal{E}_1, \varphi_2 \approx \psi_2 \in \mathcal{E}_2.$$

Now all we need is: $\mathbb{SL} \models (\varphi \approx \psi) \Leftrightarrow (\varphi \leftrightarrow \psi) \geq \mathbf{1}$

Outline

First, the simple case

Theorem (C–Noguera. Studia Logica, 2013)

Let L *be a substructural logic with an axiomatic system having rules* Ru *and let* $\nabla(p, q, \overrightarrow{r})$ *be a set of formulas such that*

$$\varphi \vdash_{\mathrm{L}} \varphi\nabla\psi \qquad \psi \vdash_{\mathrm{L}} \varphi\nabla\psi \qquad \psi\nabla\varphi \vdash_{\mathrm{L}} \varphi\nabla\psi \qquad \varphi\nabla\varphi \vdash_{\mathrm{L}} \varphi$$

Then ∇ *is a p-disjunction in* L *iff for each* χ *and each* $T \rhd \varphi \in$ Ru*:*

$$\{\psi\nabla\chi \mid \psi \in T\} \vdash_{\mathrm{L}} \varphi\nabla\chi$$

Corollary

Let L_0 *be a substructural logic with a p-disjunction* ∇ *and let* $\mathcal{L}$ *be axiomatized by adding rules* Ru *to any axiomatic system of* L_0*.*
Then ∇ *is a p-disjunction in* L *iff for each* χ *and each* $T \rhd \varphi \in$ Ru*:*

$$\{\psi\nabla\chi \mid \psi \in T\} \vdash_{\mathrm{L}} \varphi\nabla\chi$$

Second, a bit more tricky

Let us consider the following rules:

(MP)	$\varphi, \varphi \to \psi \rhd \psi$		*modus ponens*
(Adj)	$\varphi \rhd \varphi \wedge \mathbf{1}$		unit adjunction
(PN)	$\varphi \rhd \lambda_\alpha(\varphi)$	$\varphi \rhd \rho_\alpha(\varphi)$	product normality

where

- a left conjugate of φ is $\lambda_\alpha(\varphi) = (\alpha \setminus \varphi \,\&\, \alpha) \wedge \mathbf{1}$
- a right conjugate of φ is $\rho_\alpha(\varphi) = (\alpha \,\&\, \varphi \,/\, \alpha) \wedge \mathbf{1}$

Theorem (Folklore)

Logic	The only rules needed in its axiomatization
$\mathrm{FL_{ew}}$	modus ponens
$\mathrm{FL_{e}}$	modus ponens *and unit adjunction*
FL	modus ponens *and product normality*

What about SL?

We need more conjugates:

$$\alpha_{\delta,\varepsilon}(\varphi) = (\delta \mathbin{\&} \varepsilon \setminus \delta \mathbin{\&} (\varepsilon \mathbin{\&} \varphi))$$
$$\alpha'_{\delta,\varepsilon}(\varphi) = (\delta \mathbin{\&} \varepsilon \setminus (\delta \mathbin{\&} \varphi) \mathbin{\&} \varepsilon)$$
$$\beta_{\delta,\varepsilon}(\varphi) = (\delta \setminus (\varepsilon \setminus (\varepsilon \mathbin{\&} \delta) \mathbin{\&} \varphi)$$
$$\beta'_{\delta,\varepsilon}(\varphi) = (\delta \setminus ((\delta \mathbin{\&} \varepsilon) \mathbin{\&} \varphi \,/\, \varepsilon)$$

And rules of the form:

$$\varphi \rhd \eta_{\delta,\varepsilon}(\varphi)$$

for $\eta \in \{\alpha, \alpha', \beta, \beta'\}$

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$$\beta'_{\delta,\varepsilon}(\varphi) = (\delta \setminus ((\delta \mathbin{\&} \varepsilon) \mathbin{\&} \varphi / \varepsilon)$$

And rules of the form:

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for $\eta \in \{\alpha, \alpha', \beta, \beta'\}$

For the proof see: C–Horčík–Noguera. Non-associative substructural logics and their semilinear extensions: Axiomatization and completeness properties. The Review of Symbolic Logic, 2013

Conventions

Let us consider a new propositional variable $\star$

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We write $\delta(\varphi)$ for a formula resulting from δ by replacing all $\star$ by φ.

Definition (Iterated Γ-formulae)

Let Γ be a set of $\star$-formulae. We define the sets of $\star$-formulae Γ^* as the smallest set s.t. :

- $\star \in \Gamma^*$,
- $\delta(\chi) \in \Gamma^*$ for each $\delta \in \Gamma$ and each $\chi \in \Gamma^*$.

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The rest of this section is based on C–Horčík–Noguera. RSL, 2013

Main definition

Definition

L is almost (MP)-based w.r.t. a set of basic deduction terms bDT if it has an axiomatic system where

- there are no rules with three or more premises
- there is only one rule with two premises: *modus ponens*
- the remaining rules are from $\{\varphi \vdash \chi(\varphi) \mid \varphi \in \mathrm{Fm}, \chi \in \mathrm{bDT}\}$
- for each $\beta \in \mathrm{bDT}$ there is $\beta' \in \mathrm{bDT}^*$ s.t.:

$$\vdash_{\mathrm{L}} \beta'(\varphi \to \psi) \to (\beta(\varphi) \to \beta(\psi)).$$

Almost-Implicational Deduction Theorem

Definition (Conjuncted Γ-formulae)

Let Γ be a set of $\star$-formulae. We define the sets of $\star$-formulae $\Pi(\Gamma)$ as the smallest set containing $\Gamma \cup \{\mathbf{1}\}$ and closed under $\&$.

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \delta(\varphi) \to \psi \textit{ for some } \delta \in \Pi(\mathrm{bDT}^*).$$

Filter generation

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae:*

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Filter generation

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Let* $\boldsymbol{A}$ *be an* $\mathcal{L}$*-algebra and* $X \cup \{x\} \subseteq A$*. Then*

$$y \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X, x) \qquad \textit{iff} \qquad y \to y \in \mathrm{Fi}^{\boldsymbol{A}}_{\mathrm{L}}(X) \text{ for some } \delta \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(x).$$

$$\Gamma^{\boldsymbol{A}}(x) = \{\delta(x, a_1, \ldots, a_n) \mid \delta(\star, p_1, \ldots, p_n) \in \Gamma \text{ and } a_1, \ldots, a_n \in A\}$$

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$$\Gamma^{\boldsymbol{A}}(X) = \Pi(\bigcup\{\Gamma^{\boldsymbol{A}}(x) \mid x \in X\})$$

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$$\Gamma^{\boldsymbol{A}}(X) = \Pi(\bigcup\{\Gamma^{\boldsymbol{A}}(x) \mid x \in X\})$$

Corollary

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Let* $\boldsymbol{A}$ *be an* L*-algebra and* $X \subseteq A$*. Then*

$$\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \{a \in A \mid a \geq y \text{ for some } y \in (\Pi(\mathrm{bDT}^*))^{\boldsymbol{A}}(X)\}$$

Disjunction in almost (MP)-based logics

Theorem

Let L *be almost* (MP)*-based w.r.t. a set of basic deductive terms* bDT*. Then*

$$\nabla_{\mathrm{L}} = \{\alpha(p) \vee \beta(q) \mid \alpha, \beta \in (\mathrm{bDT} \cup \{\star \wedge \mathbf{1}\})^{*}\}$$

is a (p-)disjunction in L*.*

Outline

Semilinear logics

Let us by $\mathbb{Q}^{\ell}_{\mathrm{L}}$ denote the class of linearly ordered L-algebras.

Definition

A substructural logic L is called semilinear if

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad \{\psi \geq \mathbf{1} \mid \psi \in T\} \models_{\mathbb{Q}^{\ell}_{\mathrm{L}}} \varphi \geq \mathbf{1}$$

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This section is based on C–Horčík–Noguera. RSL, 2013

Note: some of the results hold in much wider setting.

Characterizations of substructural semilinear logics

Theorem

Let L *be a substructural logic. TFAE:*

1. L *is semilinear*
2. $\mathbb{Q}_{\mathrm{L}} = \mathbf{Q}(\mathbb{Q}^{\ell}_{\mathrm{L}})$
3. $\mathbb{Q}^{\ell}_{\mathrm{L}} = (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}$
4. *Each* L*-algebra is a subdirect product of* L*-chains*
5. *Any* L*-filter in an* $\mathcal{L}$*-algebra is an intersection of linear ones*

   a filter F is *linear* if $x \to y \in F$ or $y \to x \in F$, for each x, y
6. *The following metarule holds:*

$$\frac{T, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad T, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{T \vdash_{\mathrm{L}} \chi}$$

Characterizations of substructural semilinear logics

Theorem

Let L *be a substructural logic and an axiomatic system* $\mathcal{AX}$*. TFAE:*

1. L *is semilinear,*
2. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and enjoys the metarule:*

$$\frac{T, \varphi \vdash_{\mathrm{L}} \chi \qquad T, \psi \vdash_{\mathrm{L}} \chi}{T, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}$$

3. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and any* L*-filter in an* $\mathcal{L}$*-algebra is an intersection of* $\vee$*-prime ones,*
4. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and for every rule* $T \rhd \varphi$ *in* $\mathcal{AX}$ *and propositional variable* p *not occurring in* T, φ *we have*

$$\{\psi \vee \chi \mid \psi \in T\} \vdash_{\mathrm{L}} \varphi \vee \chi$$

Weakest semilinear extension

Theorem

- *There is the least semilinear logic extending* L, *denoted as* L^ℓ
- $\mathrm{L}^\ell = \mathrm{L}_{\mathbf{Q}(\mathbb{Q}^\ell_{\mathrm{L}})}$
- *If* L *is almost* (MP)*-based with* bDT, *then* L^ℓ *is axiomatized by adding axioms:*

$$((\varphi \to \psi) \wedge \mathbf{1}) \vee \delta((\psi \to \varphi) \wedge \mathbf{1}), \text{ for each } \delta \in \mathrm{bDT} \cup \{\star\}$$

Corollary

Let $\mathbb{Q}$ *be a class of residuated structures s.t.* $\mathrm{L}_{\mathbb{Q}}$ *is an almost* (MP)*-based with* bDT. *Then* $\mathbf{Q}(\{\boldsymbol{A} \in \mathbb{Q} \mid \boldsymbol{A} \text{ linear}\})$ *is a relative subvariety of* $\mathbb{Q}$ *axiomatized (relative to* $\mathbb{Q}$*) by*

$$((\varphi \to \psi) \wedge \mathbf{1}) \vee \delta((\psi \to \varphi) \wedge \mathbf{1}) \approx \mathbf{1}, \text{ for each } \delta \in \mathrm{bDT} \cup \{\star\}$$

Characterizations of completeness properties

Let L be substructural semilinear logic and $\mathbb{K}$ a class of L-chains.

Theorem (Characterization of strong $\mathbb{K}$-completeness)

1. *For each $T \cup \{\varphi\}$ holds:* $T \vdash_{\mathrm{L}} \varphi$ *iff* $\{\psi \geq \mathbf{1} \mid \psi \in T\} \models_{\mathbb{K}} \varphi \geq \mathbf{1}$.
2. $\mathbb{Q}_{\mathrm{L}} = \mathbf{ISP}_{\sigma\text{-}f}(\mathbb{K})$.
3. *Each countable L-chain is embeddable into some member of $\mathbb{K}$.*

Theorem (Characterization of finite strong $\mathbb{K}$-completeness)

1. *For each finite $T \cup \{\varphi\}$ holds:* $T \vdash_{\mathrm{L}} \varphi$ *iff* $\{\psi \geq \mathbf{1} \mid \psi \in T\} \models_{\mathbb{K}} \varphi \geq \mathbf{1}$.
2. $\mathbb{Q}_{\mathrm{L}} = \mathbf{Q}(\mathbb{K})$, *i.e., $\mathbb{K}$ generates $\mathbb{Q}_{\mathrm{L}}$ as a quasivariety.*
3. *Each finite subset of any L-chain is partially embeddable into an element of $\mathbb{K}$.*

Finite chain semantics

Let $\mathcal{F}$ be a class of finite chains

Theorem (Characterization of strong finite-chain completeness)

1. L *enjoys the* $\mathrm{S}\mathcal{F}\mathrm{C}$,
2. *All* L*-chains are finite,*
3. *There exists* $n \in \mathbb{N}$ *such each* L*-chain has at most* n *elements,*
4. *There exists* $n \in \mathbb{N}$ *such that* $\emptyset \vdash_{\mathrm{L}} \bigvee_{i<n}(x_i \to x_{i+1})$.

Known results: $\mathrm{FS}\mathcal{F}\mathrm{C}$ fails in FL^{ℓ} and $\mathrm{FL}^{\ell}_{\mathrm{e}}$
$\mathrm{FS}\mathcal{F}\mathrm{C}$ holds in $\mathrm{FL}^{\ell}_{X\cup\{\mathrm{w}\}}$ and SL^{ℓ}_{X}

Open problems: $\mathrm{FS}\mathcal{F}\mathrm{C}$ of $\mathrm{FL}^{\ell}_{\mathrm{c}}$ and $\mathrm{FL}^{\ell}_{\mathrm{ec}}$

Standard completeness

Let $\mathcal{R}$ be a class of chains with domain ((half)-open) real unit interval with usual lattice order

Known results: $\mathrm{FS}\mathcal{R}\mathrm{C}$ fails in FL^ℓ and $\mathrm{FL}^\ell_\mathrm{c}$
$\mathrm{S}\mathcal{R}\mathrm{C}$ holds in $\mathrm{FL}^\ell_\mathrm{e}$, $\mathrm{FL}^\ell_\mathrm{w}$, $\mathrm{FL}^\ell_\mathrm{ew}$, $\mathrm{FL}^\ell_\mathrm{w,c}$ and SL^ℓ_X
$\mathrm{S}\mathcal{R}\mathrm{C}$ fails but $\mathrm{FS}\mathcal{R}\mathrm{C}$ holds in logic of BL- and MV-alg.

Open problems: (F)S$\mathcal{R}$C of $\mathrm{FL}^\ell_\mathrm{ec}$

Outline

There is a nice bridge between logic and algebra . . .

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