Reticulations on residuated lattices

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A map $\lambda : A \to L$ is called *reticulation* if it satisfies

(R1) $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$

(R2) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$

(R3) $\lambda(0) = 0, \ \lambda(1) = 1$

(R4) $\lambda : A \to L$ **is onto**

(R5) $\lambda(a) \le \lambda(b) \Longleftrightarrow a^n \le b$ **for some** $n \in N$

Then we have

1. $\mathrm{Spec}(A)$ **is a compact** T_0**-space**

2. $\mathrm{Spec}(A)$ **and** $\mathrm{Spec}(L)$ **are homeomorphic**

$$\mathrm{Spec}(A) \cong \mathrm{Spec}(L)$$

Results

1. **A reticulation map λ can be defined by two conditions:**

   (R4) $\lambda : A \to L$ **is onto;**

   (R5) $\lambda(a) \leq \lambda(b) \Longleftrightarrow (\exists n \in N)$ **s.t.** $a^n \leq b$

2. **A reticulation map λ can be considered as a lattice homomorphism between A and L. Moreover, $(\mathcal{PF}(A), \xi)$ is a unique reticulation on A up to isomorphism and**

$$A/\operatorname{ker}(\lambda) \cong \mathcal{PF}(A)$$

An algebraic system $(A, \wedge, \vee, \odot, \to, 0, 1)$ **is called a** *residuated lattice* **if**

(1) $(A, \wedge, \vee, 0, 1)$ **is a bounded lattice;**

(2) $(A, \odot, 1)$ **is a commutative monoid;**

(3) For all $a, b, c \in A$**,**

$$a \odot b \leq c \Longleftrightarrow a \leq b \to c.$$

Proposition 1 **Let A be a residuated lattice. For all $a, b, c \in A$, we have**

(1) $a \leq b \Longleftrightarrow a \to b = 1$

(2) $a \to (b \to c) = a \odot b \to c = b \to (a \to c)$

(3) $a \odot (a \to b) \leq b$

(4) $a \to b \leq (b \to c) \to (a \to c)$

(5) $a \to b \leq (c \to a) \to (c \to b)$

$F \subseteq A$ $(F \neq \emptyset)$ **is called a** *filter* $(F \in \mathcal{F}(A))$ **if**

$\Longleftrightarrow$ **(F1)** $x, y \in F$ **implies** $x \odot y \in F$**;**

(F2) If $x \in F$ **and** $x \leq y$**, then** $y \in F$**.**

$$[a) = \{b \in A \,|\, (\exists n \in N) \text{ s.t. } a^n \leq b\} \quad (a \in A)$$

is called a *principal* **filter** $([a) \in \mathcal{PF}(A))$

A filter $P(\neq A)$ **is called** *prime* $(P \in \mathrm{Spec}(A))$ **if it satisfies the condition**

$$a \vee b \in P \text{ **implies** } a \in P \text{ **or** } b \in P$$

For a lattice L, a subset $F \subseteq L$ $(F \neq \emptyset)$ is called a *lattice filter* $(F \in \mathcal{LF}(L))$ **if**

(LF1) $x, y \in F \Rightarrow x \wedge y \in F$

(LF2) $x \in F$ **and** $x \leq y \Rightarrow y \in F$

Moreover, $F \in \mathcal{FL}(L), F(\neq L)$ **is called** *prime* $(F \in \text{Spec}(L))$ **if it satisfies the condition**

$$x \vee y \in F \Rightarrow x \in F \text{ or } y \in F$$

Let A be a residuated lattice and L a bounded distributive lattice. For any subset $S \subseteq A, T \subseteq L$, we take

$$D_A(S) = \{P \in \mathsf{Spec}(A) \mid S \not\subseteq P\}$$
$$D_L(T) = \{P \in \mathsf{Spec}(L) \mid T \not\subseteq P\}$$

Proposition 2

(1) $\tau_A = \{D_A(S) \mid S \subseteq A\}$ **is a topology on** $\mathsf{Spec}(A)$

(2) $\sigma_L = \{D_L(T) \mid T \subseteq L\}$ **is a topology on** $\mathsf{Spec}(L)$

A pair (L, λ) of a bounded distributive lattice L and a map $\lambda : A \to L$ is called a ***reticulation*** **on A if**

(R1) $\lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$**;**

(R2) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$**;**

(R3) $\lambda(0) = 0, \ \lambda(1) = 1$**;**

(R4) $\lambda : A \to L$ **is onto;**

(R5) $\lambda(a) \leq \lambda(b) \Longleftrightarrow a^n \leq b$ **for some $n \in N$.**

Proposition 3 Let (L, λ) be a reticulation on A. Then we have

(1) $a \leq b \;\Rightarrow\; \lambda(a) \leq \lambda(b)$

(2) $\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$

(3) For all $n \in N$, $\lambda(a^n) = \lambda(a)$

(4) $\lambda(a) = \lambda(b) \Longleftrightarrow [a) = [b)$

Theorem 4 **[Mureșan] Let (L, λ) be a reticulation on A. Then**

(a) $\mathrm{Spec}(A)$ **and** $\mathrm{Spec}(L)$ **are topological spaces.**

(b) $\lambda^* : \mathrm{Spec}(L) \to \mathrm{Spec}(A)$ **is a homeomorphism, where** $\lambda^*(P) = \lambda^{-1}(P)$ $(P \in \mathrm{Spec}(L))$**.**

(c) For two reticulations (L_1, λ_1) **and** (L_2, λ_2) **on** A**, there exists a lattice isomorphism** $f : L_1 \to L_2$ **such that** $\lambda_1 \circ f = \lambda_2$**.**

(d) $(\mathcal{PF}(A), \eta)$ **is a reticulation on** A**, where** $\eta(a) = [a)$**.**

Let $f : A \to L$ be a map with (R4) and (R5).

(R4) $f : A \to L$ **is onto;**

(R5) $f(a) \leq f(b) \Longleftrightarrow (\exists n \in N)$ **s.t.** $a^n \leq b$

Lemma 5

(1) $a \leq b \Rightarrow f(a) \leq f(b)$

(2) $f(a \wedge b) = f(a \odot b)$

(3) $f(a \wedge b) = f(a) \wedge f(b) \cdots$(R1)

(4) $f(a \vee b) = f(a) \vee f(b) \cdots$(R2)

(5) $f(0) = 0,\ f(1) = 1 \cdots$(R3)

Theorem 6 **A map $f : A \to L$ is a reticulation on A if and only if**

(R4) $f : A \to L$ **is onto;**

(R5) $f(a) \leq f(b) \iff (\exists n \in N)$ **s.t.** $a^n \leq b$

Let (L, λ) be a reticulation on A. Then we have

(h1) $\lambda(0) = 0, \ \lambda(1) = 1$

(h2) $\lambda(a \wedge b) = \lambda(a \odot b) = \lambda(a) \wedge \lambda(b)$

(h3) $\lambda(a \vee b) = \lambda(a) \vee \lambda(b)$

$$\Downarrow$$

$\lambda : A \to L$ **is a onto lattice homomorphism.**

If we take $\ker(\lambda) = \{(a, b) \,|\, \lambda(a) = \lambda(b), a, b \in A\}$**, then**

Proposition 7 $\ker(\lambda)$ **is a congruence on** A **w.r.t.** $\wedge, \odot, \vee$**.**

$$a/\ker(\lambda) = \{b \in A \mid (a,b) \in \ker(\lambda)\}$$

$$A/\ker(\lambda) = \{a/\ker(\lambda) \mid a \in A\}$$

We define $\sqcap, \sqcup, \mathbf{0}, \mathbf{1}$ on $A/\ker(\lambda)$ as follows:

$$\begin{aligned} a/\ker(\lambda) \sqcap b/\ker(\lambda) &= (a \wedge b)/\ker(\lambda) \\ &= (a \odot b)/\ker(\lambda) \\ a/\ker(\lambda) \sqcup b/\ker(\lambda) &= (a \vee b)/\ker(\lambda) \\ \mathbf{0} &= 0/\ker(\lambda), \\ \mathbf{1} &= 1/\ker(\lambda) \end{aligned}$$

Theorem 8 **[Homomorphism Theorem]**

(1) $(A/\ker(\lambda), \sqcap, \sqcup, 0, 1)$ **is a bounded distributive lattice.**

(2) A map $\nu : A \to A/\ker(\lambda)$ **defined by** $\nu(a) = a/\ker(\lambda)$ **gives a reticulation** $(A/\ker(\lambda), \nu)$ **on** A **and**

$$A/\ker(\lambda) \cong L.$$

Let $D_A(a) = \{P \in \mathsf{Spec}(A) \,|\, a \notin P\}$**. If we define a relation** $\equiv$ **on** A **by** $a \equiv b \Longleftrightarrow D_A(a) = D_A(b)$**, then** $\equiv$ **is a congruence on** A**.**

Theorem **[Mureșan] For** $A/_{\equiv} = \{[a] \,|\, a \in A\}$**, we have**

1. $(A/_{\equiv}, \wedge, \vee, [0], [1])$ **is a bounded distributive lattice.**

2. $(A/_{\equiv}, \eta)$ **is a reticulation on** A**.**

If we note that $\lambda(a) = \lambda(b) \iff [a) = [b)$, then

$$
\begin{aligned}
a \equiv b &\iff D_A(a) = D_A(b) \\
&\iff a \notin P \text{ iff } b \notin P \ (\forall P \in \mathrm{Spec}(A)) \\
&\iff a \in P \text{ iff } b \in P \ (\forall P \in \mathrm{Spec}(A)) \\
&\iff [a) = [b) \\
&\iff \lambda(a) = \lambda(b) \\
&\iff (a, b) \in \ker(\lambda)
\end{aligned}
$$

This implies that the binary relation $\equiv$ is identical with the kernel $\ker(\lambda)$ of λ.

If we define an order $\sqsubseteq$ on $\mathcal{PF}(A)$ by

$$[a) \sqsubseteq [b) \Longleftrightarrow [b) \subseteq [a),$$

then $\mathcal{PF}(A)$ is a bounded distributive lattice.

We define a map $\xi : A \to \mathcal{PF}(A)$ by $\xi(a) = [a)$.

Theorem 9 $(\mathcal{PF}(A), \xi)$ **is the reticulation of A and**

$$A/\ker(\lambda) \cong \mathcal{PF}(A).$$

$$\Uparrow$$

Homomorphism Theorem