Representing integral quantales and residuated lattices by tolerances

by Sándor Radeleczki, Math. Institute, Univ. of Miskolc (joint research with Kalle Kaarli, Tartu Univ.)

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Quantales are certain partially ordered algebraic structures that generalize locales (point free topologies) as well as various multplicative lattices of ideals from ring theory.
Residuated lattices were introduced by Dilworth and Ward and they are used in several branches of mathematics, including areas of ideal lattices of rings, lattice-ordered groups, multivalued logic and formal languages.

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In our lecture, we try to show that integral quantales and complete integral residuated lattices are strongly related with the **complete tolerances** of their underlying lattice.

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A **quantale** is an algebraic structure $\mathbb{Q} = (L, \vee, \odot)$, such that $(L, \leq)$ is a complete lattice (induced by the join operation $\vee$) and $(L, \odot)$ is a semigroup satisfying

$$a \odot \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \odot b_i) \text{ and } \left(\bigvee_{i \in I} b_i \right) \odot a = \bigvee_{i \in I} (b_i \odot a).$$

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for all $a \in L$ and $b_i \in L$, $i \in I$. $\mathbb{Q}$ is called commutative, if $\odot$ is commutative, and $\mathbb{Q}$ is unital, whenever $(L, \odot)$ is a monoid. A unital quantale in which the neutral element of $\odot$ coincides to the greatest element 1 of the lattice L is called integral. Hence in any integral quantale

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A subset $K \subseteq L$ is called a subquantale of $\mathbb{Q}$ if it is closed under arbitrary joins and $\odot$.

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(b) The two-sided ideals of a ring $(R, +, \cdot)$ with unit form an integral quantale $(\mathcal{I}(R), \vee, \bullet)$, where $(\mathcal{I}(R), \vee, \cap)$ is the complete lattice of the ideals of R and $\bullet$ is their usual multiplication, i.e. for any $I, J \in \mathcal{I}(R)$ we have $I \bullet J := \{\sum_{\text{fin}} i \cdot j \mid i \in I, j \in J\}$.

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A **residuated lattice** is an algebra $\mathcal{L} = (L, \vee, \wedge, \odot, \backslash, /, 1)$ of type (2,2,2,2,2,0) such that

(i) $(L, \vee, \wedge)$ is a lattice,

(ii) $(L, \odot)$ is a semigroup satisfying $1 \odot x = x \odot 1 = x$, for all $x \in L$.

(iii) $\mathcal{L}$ satisfies the *adjointness* properties, that is, for all $x, y, z \in L$

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$\mathcal{L}$ is called commutative, if $\odot$ is commutative. In this case $x \backslash y$ and y/x being equal, they are denoted as $x \to y$.

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Example 2.

Consider the integral quantale $(\mathcal{I}(R), \vee, \bullet)$ from Example 1. This induces a complete integral residuated lattice $(\mathcal{I}(R), \cap, \vee, \bullet, \backslash, /, R)$, where the operations $\backslash$, $/$ are defined as follows:
$I \backslash J = \{r \in R \mid I \cdot r \subseteq J\}$, $J/I = \{r \in R \mid r \cdot I \subseteq J\}$.

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Indeed, we have $I \bullet J \subseteq K \Leftrightarrow J \subseteq I\backslash K \Leftrightarrow I \subseteq K/J$.

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$$\left(\bigwedge_{i\in I} a_i, \bigwedge_{i\in I} b_i\right) \in T \text{ and } \left(\bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i\right) \in T.$$

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The set of complete tolerances of L is denoted by $\mathrm{CTol}(L)$. $\mathrm{CTol}(L)$ is a complete lattice with respect to $\subseteq$, with least element $\triangle = \{(x, x) \mid x \in L\}$ and greatest element $\nabla = L \times L$.

If S and T are two complete tolerances of L, then let $S \circ T$ stand for their relational product. The symmetrized product of S and T is defined as follows:

$$S * T = \{(x, y) \in L^2 \mid (x \wedge y, x \vee y) \in S \circ T\}. \qquad (1.3)$$

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Lemma 1.

For arbitrary complete tolerances S and T_i, $i \in I$ of a complete lattice L,

$$\left(\bigcap\{T_i \mid i \in I\}\right) * S = \bigcap\{T_i * S \mid i \in I\}, \text{ and}$$
$$S * \left(\bigcap\{T_i \mid i \in I\}\right) = \bigcap\{S * T_i \mid i \in I\}.$$

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Now, let $(\mathrm{CTol}(L), \cap, \vee)$ stand for the dual of the lattice $(\mathrm{CTol}(L), \vee, \cap)$ (hence its greatest element is $\triangle$.) Since $T * \triangle = \triangle * T = T$, $(\mathrm{CTol}(L), *)$ is a semigroup with unit element $\triangle$, whence we obtain:

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Clearly, for all $S, T \in \mathrm{CTol}(L)$ we can define the operations:

$$S \backslash T = \bigcap\{Z \in \mathrm{CTol}(L) \mid S * Z \supseteq T\} \text{ and } S/T = \bigcap\{Z \in \mathrm{CTol}(L) \mid Z * S \supseteq T\}$$

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We note that in [BK] it was also shown that on lattice $(\mathrm{CTol}(L), \cap, \vee)$ can be defined a complete integral residuated lattice.
The corresponding integral quantale was $(\mathrm{CTol}(L), \cap, \otimes)$, and the binary operation $\otimes$ was defined as follows:

$$S \otimes T = (S \circ \ \geq \circ T) \cap (T \circ \ \leq \circ S).$$

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Important notation. If $T \subseteq L^2$ is a complete tolerance of $(L, \vee, \wedge)$, then for any $a \in L$ we define:

$$a_T := \bigwedge \{z \in L \mid (a, z) \in T\} \text{ and } a^T := \bigvee \{z \in L \mid (a, z) \in T\}.$$

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Conversely, if $\lambda\colon L \to L$, $\mu\colon L \to L$ is an adjoint pair of maps such that $\lambda(x) \leq x$ (and equivalently, $x \leq \mu(x)$), for all $x \in L$, then

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Denote by $\mathrm{Cdend}(L)$ the complete decreasing $\vee$-endomorphism of L. $\mathrm{Cdend}(L)$ can be ordered in a natural way; for any $\rho, \sigma \in \mathrm{Cdend}(L)$,

$$\rho \leq \sigma \;\Leftrightarrow\; \rho(x) \leq \sigma(x), \text{ for all } x \in L. \qquad (2.2)$$

$(\mathrm{Cdend}(L), \leq)$ is a complete lattice, and the mapping $\lambda \mapsto T_\lambda$ is a dual lattice isomorphism between $(\mathrm{Cdend}(L), \leq)$ and $(\mathrm{CTol}(L), \subseteq)$ (see [J1] or [K]). In other words,

$$\rho \leq \sigma \Leftrightarrow T_\sigma \subseteq T_\rho, \tag{2.3}$$

for all $\rho, \sigma \in \mathrm{Cdend}(L)$. Moreover, λ is idempotent $(\lambda \circ \lambda = \lambda)$ iff T_λ is transitive, i.e. it is a *complete congruence* of L (cf. [J2]).

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Our starting observation

Let $\mathbb{Q} = (L, \vee, \odot)$ be an integral quantale and $\mathcal{L} = (L, \vee, \wedge, \odot, \backslash, /, 0, 1)$ the complete integral residuated lattice induced by $\mathbb{Q}$. Then for any $a \in L$ the map $\lambda_a(x) = a \odot x$, $x \in L$ is a complete $\vee$-endomorphism, and $\mu_a(x) = a \backslash x$, $x \in L$ is a complete $\wedge$-endomorphism, and they form a residuated pair, i.e:

$$\lambda_a(x) \leq y \Leftrightarrow a \odot x \leq y \Leftrightarrow x \leq a \backslash y \Leftrightarrow x \leq \mu_a(y).$$

Since $\lambda_a(x) = a \odot x \leq x$, they determine a complete tolerance of the lattice L as follows:

$$T_a := \{(x, y) \in L^2 \mid a \odot (x \vee y) \leq x \wedge y\}. \tag{2.4}$$

The maps λ_a, $a \in L$ are called the *left translations* of the monoid $(L, \odot)$. The *right translations* of $(L, \odot)$ are the maps $\tau_a(x) = x \odot a$, $a \in L$.

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3. The main construction

Denote by Σ the set of right translations of $(L, \odot)$, i.e. $\Sigma = \{\tau_a \mid a \in L\}$, and consider the algebra $\mathcal{A} = (L; \vee, \wedge, \Sigma, 0, 1)$. Let $\mathrm{CT}(\mathcal{A})$ stand for the set of all complete tolerances of $\mathcal{A}$, i.e. the set of those elements of $\mathrm{CTol}(L)$ which are preserved by each $\tau_a \in \Sigma$.

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It is easy to see that $(\mathrm{CT}(\mathcal{A}), \subseteq)$ is a complete $\cap$-subsemilattice of $(\mathrm{CTol}(L), \subseteq)$, hence $(\mathrm{CT}(\mathcal{A}), \subseteq)$ is a complete lattice. Since $\mathrm{CT}(\mathcal{A})$ is not a complete sublattice of $\mathrm{CTol}(L)$ in general, the join in $(\mathrm{CT}(\mathcal{A}), \subseteq)$ will be denoted by $\sqcup$.

Lemma 2.

Let $\mathbb{Q} = (L, \vee, \odot)$ *be an integral quantale defined on the lattice* $(L, \vee, \wedge)$. *Then* $(\mathrm{CT}(\mathcal{A}), \cap, *)$ *is an integral subquantale of* $(\mathrm{CTol}(L), \cap, *)$, *and* $(\mathrm{CT}(\mathcal{A}), \cap, \sqcup, *, \backslash, /, \bigtriangledown, \bigtriangleup)$ *is a complete integral residuated lattice, where for every* $S, T \in \mathrm{CT}(\mathcal{A})$ *the operations* $\backslash$ *and* $/$ *are defined as*

$$S \backslash T = \bigcap \{Z \in \mathrm{CT}(\mathcal{A}) \mid S * Z \supseteq T\} \text{ and } \tag{3.1.a}$$

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For any $a, b \in L$ the least tolerance $T \in \mathrm{CT}(\mathcal{A})$ containing (a, b) is denoted by $T_{\mathcal{A}}(a, b)$ and is called the *principal complete tolerance of* $\mathcal{A}$ *corresponding to* (a, b).

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Proposition 2.

Let $\mathbb{Q} = (L, \vee, \odot)$ *be an integral quantale. Then for any* $a \in L$ *we have*

$$T_a = T_{\mathcal{A}}(a, 1).$$

Now, let $\mathrm{CPT}(\mathcal{A}) = \{ T_a \mid a \in L \}$, and consider the ordered set $(\mathrm{CPT}(\mathcal{A}), \supseteq)$. Clearly, $\mathrm{CPT}(\mathcal{A})$ is a subset of $\mathrm{CT}(\mathcal{A})$, and $\triangle = T_1 \in \mathrm{CPT}(\mathcal{A})$ and $\nabla = T_0 = T_{\mathcal{A}}(0,1) \in \mathrm{CPT}(\mathcal{A})$. Moreover, we have the following

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(i)$(\mathrm{CPT}(\mathcal{A}), \cap, *)$ *is an integral subquantale of* $(\mathrm{CT}(\mathcal{A}), \cap, *)$ *and of* $(\mathrm{CTol}(L), \cap, *)$.
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Our next results were proved by using the properties of the quantale $\mathrm{CPT}(\mathcal{A})$.

4. Main Results

Theorem 1.

(i) *Any integral quantale* $(L, \vee, \odot)$ *is isomorphic to* $(\mathrm{CPT}(\mathcal{A}), \cap, *)$, *and any complete integral residuated lattice* $\mathcal{L} = (L, \vee, \wedge, \odot, \backslash, /, 0, 1)$ *is isomorphic to* $(\mathrm{CPT}(\mathcal{A}), \cap, \sqcup, *, \backslash, /, \triangledown, \triangle)$. *In particular, we have* $T_{a \backslash b.} = T_a \backslash T_b$ and $T_{a/b} = T_a / T_b$.

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(ii) *If* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is commutative, then* $\Phi \colon L \rightarrow \mathrm{CPT}(\mathcal{A})$, $\Phi(a) = T_a$, *for all* $a \in L$ *is an embedding of the dual of* $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *into* $(\mathrm{CT}(\mathcal{A}), \cap, \sqcup, *, \backslash, /, \bigtriangledown, \bigtriangleup)$.

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Theorem 2.

Let L *be a complete lattice. Then there exists an operation* $\odot$ *on* L *such that* $(L, \vee, \odot)$ *is an integral quantale if and only if there exists an subquantale* $(\mathcal{K}, \cap, *)$ *of* $(\mathrm{CTol}(L), \cap, *)$ *with* $\triangle \in \mathcal{K}$ *such that* $(L, \leq)$ *is dually isomorphic to* $(\mathcal{K}, \subseteq)$.

Corollary 3.

Let L be a complete lattice. L admits an integral residuated lattice structure if and only if there exists a subquantale $(\mathcal{K}, \cap, *)$ *of* $(\mathrm{CTol}(L), \cap, *)$ *with* $\triangle \in \mathcal{K}$*, such that* $(L, \leq)$ *is dually isomorphic to* $(\mathcal{K}, \subseteq)$*.*

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Properties of the underlying lattice

As an application of Theorem 1, we can deduce some properties of the underlying lattice of finite integral quantales and finite integral residuated lattices.

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Definitions 3.

(a) A lattice L with 0 is called pseudocomplemented if for each $x \in L$ there exists an $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leq x^*$.
(b) A bounded lattice L is called 0-modular, (1-modular) if L has no pentagon sublattice N_5 that contains 0 (1, respectively).
(c) A pair $a, b \in L$ is called a distributive pair if $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$ holds for any $c \in L$. The dual notion is a dually distributive pair.

Theorem 3.

Let $\mathbb{Q} = (L, \vee, \odot)$ *be a finite integral quantale. Then* $(L, \vee, \wedge)$ *is a dually pseudocomplemented and* 1*-modular lattice, and any pair of elements* $a, b \in L$ *with* $a \vee b = 1$ *is a dually distributive pair in* L*.*

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Corollary 4.

The underlying lattice of any finite integral residuated lattice $\mathcal{L} = (L, \vee, \wedge, \odot, \backslash, /, 0, 1)$ is dually pseudocomplemented and 1-modular, and any $a, b \in L$ with $a \vee b = 1$ is a dually distributive pair in it.

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