The many facets of the representation theory of residuated structures

Thomas Vetterlein

Department of Knowledge-Based Mathematical Systems,
Johannes Kepler University (Linz, Austria)

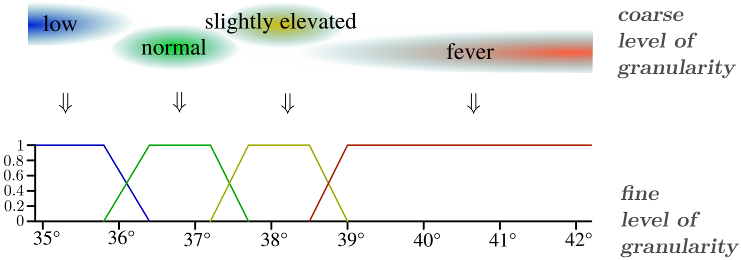
4 September 2016

JOHANNES KEPLER UNIVERSITY LINZ

MATHEMATIK JKU LINZ

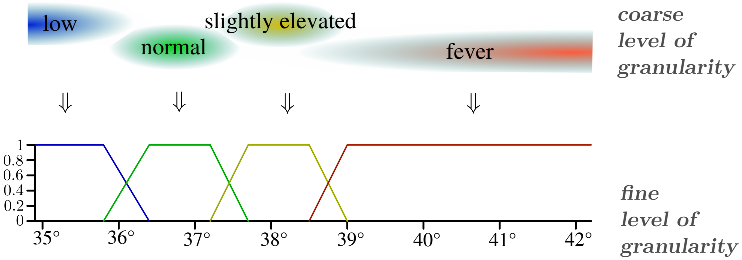
Our starting point

Fuzzy set theory offers a means of modelling natural-language expressions like *"warm"/"cold"*, *"low"/"high"*, and the like.



Our starting point

Fuzzy set theory offers a means of modelling natural-language expressions like *"warm"/"cold"*, *"low"/"high"*, and the like.



The intention is to merge different levels of granularities.

Logic based on fuzzy sets

On the fine level, properties are modelled by subsets.
Properties now being modelled by fuzzy sets,
how should we realise logical combinations?

Logic based on fuzzy sets

On the fine level, properties are modelled by subsets.
Properties now being modelled by fuzzy sets,
how should we realise logical combinations?

The common principle:

Operations on fuzzy sets are defined **pointwise**.

Logic based on fuzzy sets

On the fine level, properties are modelled by subsets.
Properties now being modelled by fuzzy sets,
how should we realise logical combinations?

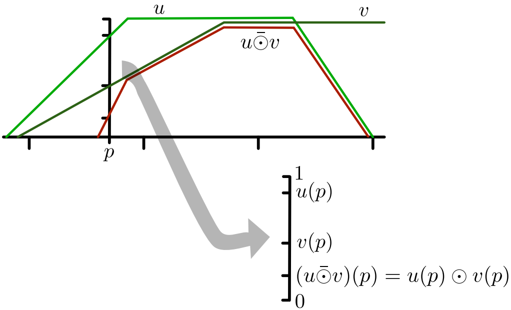
The common principle:

Operations on fuzzy sets are defined **pointwise**.

That is, there is supposed to be a binary operation

$$\odot\colon [0,1]^2 \to [0,1]$$

interpreting the conjunction:



Triangular norms

Definition (Schweizer, Sklar)

An operation $\odot\colon [0,1]^2 \to [0,1]$ is called a t-norm if:

(N1) $\odot$ is associative;

(N2) $\odot$ is commutative;

(N3) $a \odot 1 = a$;

(N4) $\odot$ is monotone:
$a \leqslant b$ implies $a \odot c \leqslant b \odot c$.

Triangular norms

Definition (Schweizer, Sklar)

An operation $\odot\colon [0,1]^2 \to [0,1]$ is called a t-norm if:

(N1) $\odot$ is associative;

(N2) $\odot$ is commutative;

(N3) $a \odot 1 = a$;

(N4) $\odot$ is monotone:
$a \leqslant b$ implies $a \odot c \leqslant b \odot c$.

The t-norm is to interpret the conjunction.

The implication is interpreted by its residuum:

$$a \to b \;=\; \max \{c \in [0,1] \colon a \odot c \leqslant b\}, \qquad a, b \in [0,1].$$

A t-norm possesses a residuum if it is left-continuous.

The logic MTL ...

- uses the real unit interval $[0, 1]$ as the set of truth degrees,
- interprets the (main) conjunction by a left-continuous t-norm and the implication by its residuum.

Mathematical fuzzy logic

The logic MTL ...

- uses the real unit interval $[0, 1]$ as the set of truth degrees,
- interprets the (main) conjunction by a left-continuous t-norm and the implication by its residuum.

MTL possesses a Hilbert-style axiomatisation (Esteva, Godo; Jenei, Montagna).

Understanding fuzzy logics?

There are some never solved issues in fuzzy logic:

- Does fuzzy logic comply with the concern of modelling natural-language expressions (of the relevant type)?

Understanding fuzzy logics?

There are some never solved issues in fuzzy logic:

- Does fuzzy logic comply with the concern of modelling natural-language expressions (of the relevant type)?
- If so, why should we choose which t-norm for which application?

Understanding fuzzy logics?

There are some never solved issues in fuzzy logic:

- Does fuzzy logic comply with the concern of modelling natural-language expressions (of the relevant type)?
- If so, why should we choose which t-norm for which application?
- Which (left-continuous) t-norms are at our disposal?

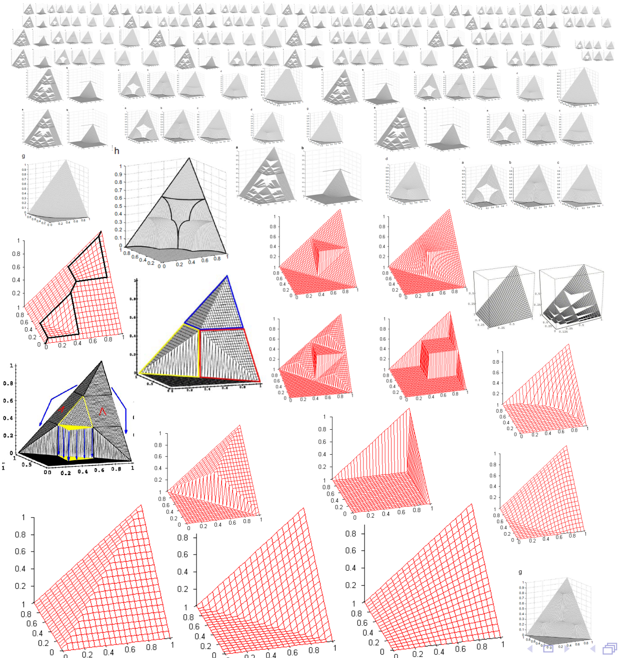
Understanding fuzzy logics?

There are some never solved issues in fuzzy logic:

- Does fuzzy logic comply with the concern of modelling natural-language expressions (of the relevant type)?
- If so, why should we choose which t-norm for which application?
- Which (left-continuous) t-norms are at our disposal?

Not being able to answer the first two questions, let us focus on the last one.

Early observation

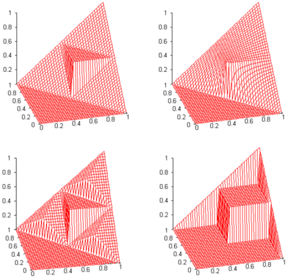


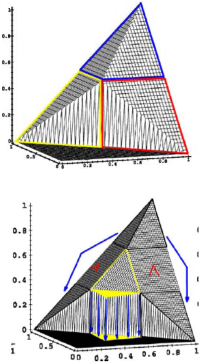
There are many t-norms.

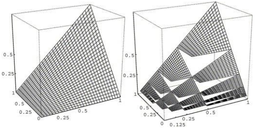
Further observation

There are also many (mainly geometric) construction methods:

- ordinal sum
- rotation and rotation-annihilation (JENEI)
- triple rotation (MAES, DE BAETS)
- H-transform (ZEMÁNKOVÁ)







Classification of left-continuous t-norms?

Our concern

Describe left-continuous t-norms systematically enough so as to make order at least out of the existing examples and their closure under the known construction methods.

Suitable ordered algebras

Definition

An ℓ-monoid is an algebra $(L; \wedge, \vee, \cdot, 1)$ such that

(1) $(L; \wedge, \vee)$ is a lattice;

(2) $(L; \cdot, 1)$ is a monoid;

(3) for any a, b, c,

$$
\begin{aligned}
a \cdot (b \vee c) &= a \cdot b \vee a \cdot c, \\
(a \vee b) \cdot c &= a \cdot c \ \vee \ b \cdot c.
\end{aligned}
$$

Suitable ordered algebras

Definition

An ℓ-monoid is an algebra $(L; \wedge, \vee, \cdot, 1)$ such that

(1) $(L; \wedge, \vee)$ is a lattice;

(2) $(L; \cdot, 1)$ is a monoid;

(3) for any a, b, c,

$$\begin{aligned} a \cdot (b \vee c) &= a \cdot b \vee a \cdot c, \\ (a \vee b) \cdot c &= a \cdot c \ \vee \ b \cdot c. \end{aligned}$$

An ℓ-monoid is called

commutative if so is $\cdot$,

integral if 1 is the top element.

Definition

A tomonoid is an ℓ-monoid whose order is total.

Definition

A tomonoid is an ℓ-monoid whose order is total.

A binary operation $\odot$ on $[0, 1]$ is a t-norm if and only if

$$([0, 1]; \wedge, \vee, \odot, 1)$$

is a commutative, integral tomonoid.

Congruences

Congruences of integral ℓ-monoids?

Congruences

Congruences of integral ℓ-monoids?

- They are not 1-regular, but we may still consider congruences induced by sub-ℓ-monoids.

Congruences

Congruences of integral ℓ-monoids?

- They are not 1-regular, but we may still consider congruences induced by sub-ℓ-monoids.
- Another type are Rees congruences: cutting off a part from below.

Filter-induced congruences

We assume in the sequel commutativity.

Definition

A filter of an integral ℓ-monoid is an upwards closed sub-ℓ-monoid.

Filter-induced congruences

We assume in the sequel commutativity.

Definition

A filter of an integral ℓ-monoid is
an upwards closed sub-ℓ-monoid.

Proposition (McCarthy; Blount, Tsinakis)

Let F be a filter of an integral ℓ-monoid L. Define, for $a, b \in L$,

$$a \, \theta_F \, b \quad \text{if } a\, f \leqslant b \text{ and } b\, f \leqslant a \text{ for some } f \in F.$$

Then θ_F is a congruence.

Filter-induced congruences

We assume in the sequel commutativity.

Definition

A filter of an integral ℓ-monoid is an upwards closed sub-ℓ-monoid.

Proposition (McCarthy; Blount, Tsinakis)

Let F be a filter of an integral ℓ-monoid L. Define, for $a, b \in L$,

$$a \, \theta_F \, b \quad \text{if } a\,f \leqslant b \text{ and } b\,f \leqslant a \text{ for some } f \in F.$$

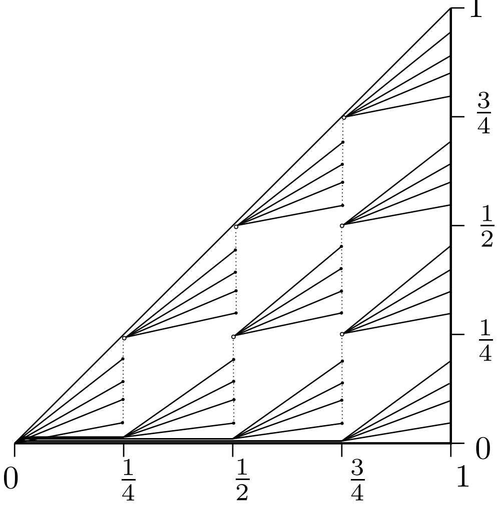
Then θ_F is a congruence.

We call the congruence classes F-classes and we denote the quotient by L/F.

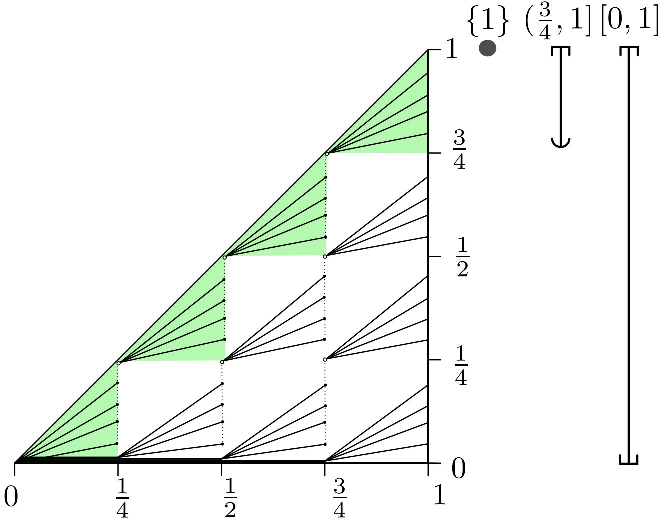
Example

Consider the integral tomonoid $([0,1]; \wedge, \vee, \odot_H, 1)$,

where $\odot_H$ is the following t-norm:

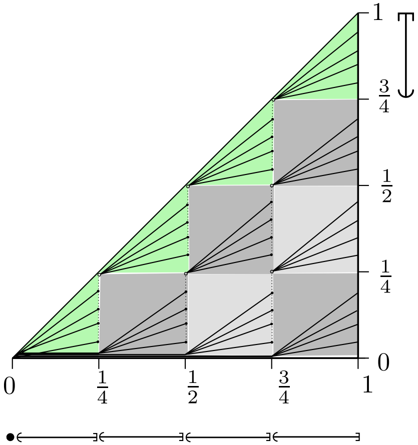


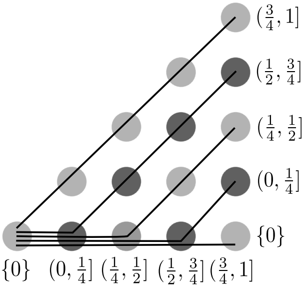
Example



The filters of $([0,1]; \wedge, \vee, \odot_H, 1)$.

Example





The quotient of $([0, 1]; \wedge, \vee, \odot_H, 1)$ by the filter $(\frac{3}{4}, 1]$.

Coextensions by ℓ-monoids

Definition

Let L be an integral ℓ-monoid and
let P be the quotient of L by a filter F.
Then we call L the coextension of P by F.

Coextensions by ℓ-monoids

Definition

Let L be an integral ℓ-monoid and
let P be the quotient of L by a filter F.
Then we call L the coextension of P by F.

Challenge

Given integral ℓ-monoids P and F,
determine the coextensions of P by F.

Idea for what follows

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

Then the product splits into the following mappings:

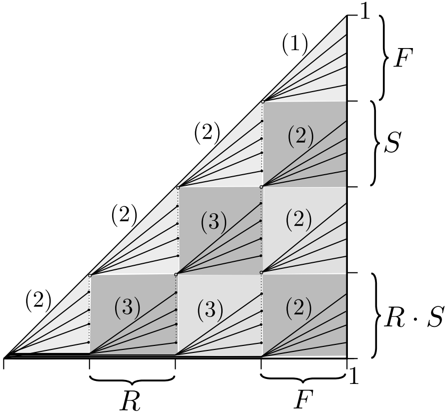
(1) $\cdot : F \times F \to F$.
(2) $\cdot : F \times R \to R$, where R is an F-class other than F.
(3) $\cdot : R \times S \to R \cdot S$, where R, S are F-classes other than F.

Idea for what follows

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

Then the product splits into the following mappings:

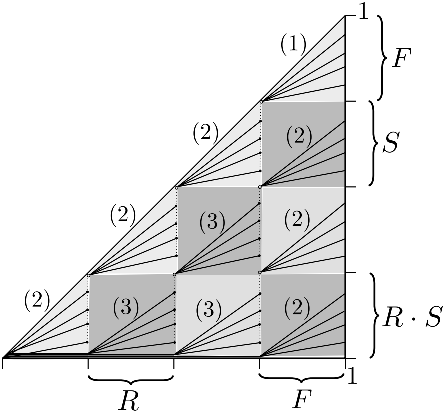
(1) $\cdot\colon F \times F \to F$.
(2) $\cdot\colon F \times R \to R$, where R is an F-class other than F.
(3) $\cdot\colon R \times S \to R \cdot S$, where R, S are F-classes other than F.



Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

Then the product splits into the following mappings:

(1) $\cdot \colon F \times F \to F$.
(2) $\cdot \colon F \times R \to R$, where R is an F-class other than F.
(3) $\cdot \colon R \times S \to R \cdot S$, where R, S are F-classes other than F.



Assume we are given L/F and F.
Our idea of how to construct L:

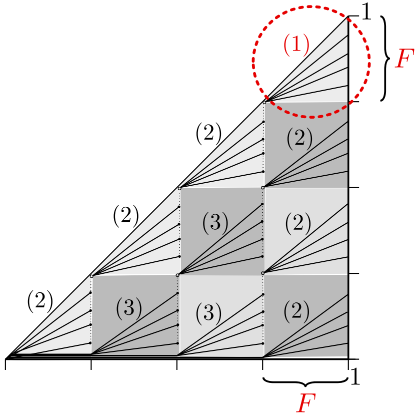
- We assume $(L; \wedge, \vee)$.
- We determine the mappings (2), (3) individually.

Part (1)

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

(1) $\cdot : F \times F \to F$.

This is the product of F and hence assumed.



Modules over an ℓ-monoid

Definition

Let F be an integral ℓ-monoid.
Then an F-module is a $\vee$-semilattice R
together with a mapping $\star\colon F \times R \to R$ such that

- $\star$ preserves (finite) joins in each argument,
- $f \star (g \star r) = f\, g \star r$ for any $r \in R$ and $f, g \in F$ and $1 \star r = r$ for any $r \in R$.

Modules over an ℓ-monoid

Definition

Let F be an integral ℓ-monoid.
Then an F-module is a $\vee$-semilattice R
together with a mapping $\star\colon F \times R \to R$ such that

- $\star$ preserves (finite) joins in each argument,
- $f \star (g \star r) = f\,g \star r$ for any $r \in R$ and $f, g \in F$
  and $1 \star r = r$ for any $r \in R$.

Call an F-module weakly transitive if,
for any $r, s \in R$ there is an $f \in F$ such that $f \star r \leqslant s$.

Definition

Let F be an integral ℓ-monoid.
Then an *F-module* is a $\vee$-semilattice R
together with a mapping $\star\colon F \times R \to R$ such that

- $\star$ preserves (finite) joins in each argument,
- $f \star (g \star r) = f\,g \star r$ for any $r \in R$ and $f, g \in F$ and $1 \star r = r$ for any $r \in R$.

Call an F-module *weakly transitive* if,
for any $r, s \in R$ there is an $f \in F$ such that $f \star r \leqslant s$.

Notes.

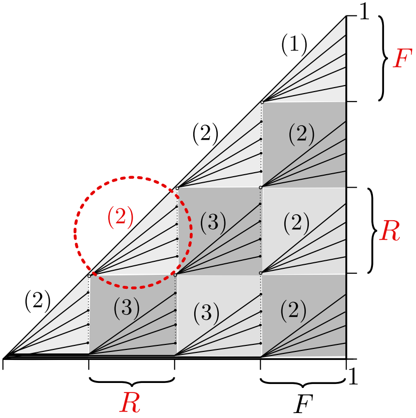
- This is the $\vee$-semilattice version of an *S-poset* (Fakhruddin).
- Replacing F by a quantale Q and "finite joins" by "arbitrary joins", this is a *Q-module* (Abramsky, Vickers).

Part (2)

Lemma

Let F be a filter of the integral ℓ-monoid L.
Then each F-class R is a weakly transitive F-module:

$$f \star r \;=\; f \cdot r, \quad f \in F,\ r \in R.$$



Homomorphisms of F-modules

Definition

Let R and S be F-modules.
Then $\varphi\colon R \to S$ is a homomorphism if
φ preserves joins and

$$\varphi(f \star r) \;=\; f \star \varphi(r)$$

for any $f \in F$ and $r \in R$.

Homomorphisms of F-modules

Definition

Let R and S be F-modules.
Then $\varphi\colon R \to S$ is a homomorphism if
φ preserves joins and

$$\varphi(f \star r) \;=\; f \star \varphi(r)$$

for any $f \in F$ and $r \in R$.

Definition

Let R, S, and T be F-modules.
Then $\psi\colon R \times S \to T$ is a bihomomorphism if
ψ preserves joins in each argument and

$$\psi(f \star r, s) \;=\; \psi(r, f \star s) \;=\; f \star \psi(r, s)$$

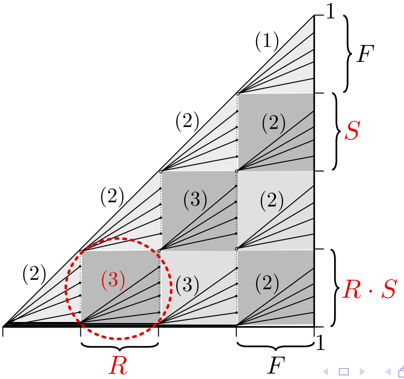
for any $f \in F$, $r \in R$, and $s \in S$.

Lemma

Let F be a filter of the integral ℓ-monoid L.
Then for any F-classes R and S, viewed as F-modules,

$$R \times S \to R \cdot S,\ (r, s) \mapsto r \cdot s$$

is a bihomomorphism.



Interlude from linear algebra

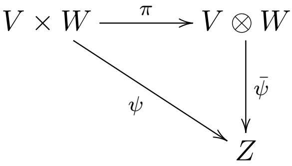
Let V, W, Z be linear spaces.
A mapping $\psi\colon V \times W \to Z$ is called bilinear
if $\psi(v, \_)$ and $\psi(\_, w)$ are linear for fixed $v \in V$, $w \in W$.

Interlude from linear algebra

Let V, W, Z be linear spaces.
A mapping $\psi\colon V \times W \to Z$ is called bilinear
if $\psi(v, \_)$ and $\psi(\_, w)$ are linear for fixed $v \in V$, $w \in W$.

The tensor product of V and W consists of
a linear space $V \otimes W$ and a bilinear map $\pi\colon V \times W \to V \otimes W$
such that:

For any bilinear map $\psi\colon V \times W \to Z$, there is a **linear** map
$\bar{\psi}\colon V \otimes W \to Z$ such that $\psi = \bar{\psi} \circ \pi$.



The construction of the tensor product:

Put $V \otimes W = F(V \times W)/N$,
where $F(V \times W)$ is the free vector space on $V \times W$
and N is the subspace spanned by

$$\begin{aligned}
&(v_1, w) + (v_2, w) - (v_1 + v_2, w),\\
&(v, w_1) + (v, w_2) - (v, w_1 + w_2),\\
&\lambda(v, w) - (\lambda v, w),\\
&\lambda(v, w) - (v, \lambda w),
\end{aligned}$$

where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and λ is a scalar.

Adapting the idea to Q-modules (Joyal, Tierney; C. Russo)

Let Q be a quantale and let R and S be Q-modules.

Then a tensor product $R \otimes_Q S$ can be defined,
and its existence proved,
similarly to the case of linear spaces.

linear spaces	Q-modules
addition	sup
scalar multiplication	action of the elements of Q

Tensor product of F-modules

Definition

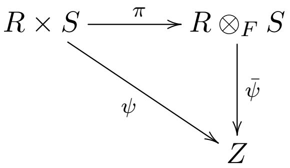
Let F be an integral ℓ-monoid and let R and S be F-modules.

A tensor product of R and S is an F-module T
together with a bihomomorphism $\pi \colon R \times S \to T$
such that:

For any bihomomorphism $\psi \colon R \times S \to U$,
there is a unique homomorphism $\bar{\psi} \colon T \to U$ such that $\psi = \bar{\psi} \circ \pi$.

In this case, (T, π) is essentially unique.

We write $R \otimes_F S$ for T, and $r \otimes_F s$ for $\pi(r, s)$.



Tensor product of F-modules

Theorem

Let F be an integral ℓ-monoid and let R and S be F-modules.

The tensor product $R \otimes_F S$ exists.

Tensor product of F-modules

Theorem

Let F be an integral ℓ-monoid and let R and S be F-modules.

The tensor product $R \otimes_F S$ exists.

The construction of the tensor product:

Put $R \otimes_F S \ = \ \mathcal{F}(R \times S)/\sim$,
where $\mathcal{F}(R \times S)$ is the free $\vee$-semilattice on $R \times S$
and $\sim$ is the smallest congruence such that

$$\begin{aligned} &\{(r_1 \vee r_2, s)\} \sim \{(r_1, s), (r_2, s)\}, \\ &\{(r, s_1 \vee s_2\} \sim \{(r, s_1), (r, s_2)\}, \\ &\{(f \star r, s)\} \sim \{(r, f \star s)\}, \end{aligned}$$

where $r, r_1, r_2 \in R,\ \ s, s_1, s_2 \in S$, and $f \in F$.

Summarising

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

Then the product splits into the following mappings:

Summarising

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

Then the product splits into the following mappings:

(1) $\cdot \colon F \times F \to F$.
This is the filter F.

Summarising

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

Then the product splits into the following mappings:

(1) $\cdot \colon F \times F \to F$.
This is the filter F.

(2) $\cdot \colon F \times R \to R$, where R is an F-class other than F.
This mapping makes R into an F-module.

Summarising

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and F its filter.

Then the product splits into the following mappings:

(1) $\cdot \colon F \times F \to F$.
This is the filter F.

(2) $\cdot \colon F \times R \to R$, where R is an F-class other than F.
This mapping makes R into an F-module.

(3) $\cdot \colon R \times S \to R \cdot S$, where R, S are F-classes other than F.
Viewing R and S as F-modules, this is a bihomomorphism and can hence be identified with a homomorphism from the tensor product $R \otimes_F S$ to $R \cdot S$.

Building a coextension

Theorem

Let P and F be integral ℓ-monoids.
Let $(L; \wedge, \vee, 1)$ be a lattice with 1 and θ a congruence of L such that $1/\theta \cong (F; \wedge, \vee, 1)$ and $L/\theta \cong (P; \wedge, \vee, 1)$.

Building a coextension

Theorem

Let P and F be integral ℓ-monoids.
Let $(L; \wedge, \vee, 1)$ be a lattice with 1 and θ a congruence of L such that $1/\theta \cong (F; \wedge, \vee, 1)$ and $L/\theta \cong (P; \wedge, \vee, 1)$.
Assume that

- each θ-class is a weakly transitive F-module;
- for each two θ-classes $R, S \neq F$, there is an F-module homomorphisms $\varphi_{RS}^{R\otimes S} \colon R \otimes S \to R \cdot S$, and we have

$$\begin{aligned} \varphi_{RST}^{RS\otimes T}(\varphi_{RS}^{R\otimes S}(r \otimes s) \otimes t) &= \varphi_{RST}^{R\otimes ST}(r \otimes \varphi_{ST}^{S\otimes T}(s \otimes t)) \\ \varphi_{R(S\vee T)}^{R\otimes(S\vee T)}(r \otimes (s \vee t)) &= \varphi_{RS}^{R\otimes S}(r \otimes s) \vee \varphi_{RT}^{R\otimes T}(r \otimes t) \end{aligned}$$

for any θ-classes $R, S, T \neq F$, and $r \in R,\ s \in S,\ t \in T$.

Building a coextension

Theorem (ctd.)

For $r, s \in L$, let

$$r \cdot s = \begin{cases} \text{the product } r \cdot s \text{ in } F & \text{if } r, s \in F, \\ r \star s & \text{if } r \in F \text{ and } s \in R \neq F, \\ s \star r & \text{if } s \in F \text{ and } r \in R \neq F, \\ \varphi_{RS}^{R \otimes S}(r \otimes s) & \text{if } r \in R \neq F \text{ and } s \in S \neq F. \end{cases}$$

Building a coextension

Theorem (ctd.)

For $r, s \in L$, let

$$r \cdot s \;=\; \begin{cases} \text{the product } r \cdot s \text{ in } F & \text{if } r, s \in F, \\ r \star s & \text{if } r \in F \text{ and } s \in R \neq F, \\ s \star r & \text{if } s \in F \text{ and } r \in R \neq F, \\ \varphi_{RS}^{R \otimes S}(r \otimes s) & \text{if } r \in R \neq F \text{ and } s \in S \neq F. \end{cases}$$

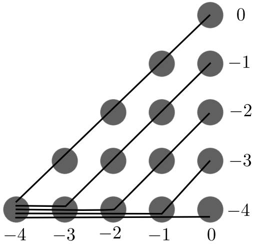
Then $(L; \wedge, \vee, \cdot, 1)$ is a coextension of P by F,
and all coextensions of P by F arise in this way.

The above example reversed

Consider the five-element Łukasiewicz chain $Ł_5 = \{-4, -3, -2, -1, 0\}$,

$$(Ł_5; \wedge, \vee, +, 0),$$

and the tomonoid $(\mathbb{R}^-; \wedge, \vee, +, 0)$.

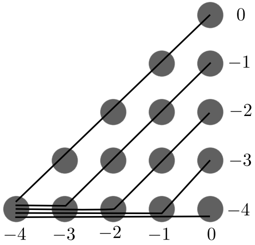


The above example reversed

Consider the five-element Łukasiewicz chain $Ł_5 = \{-4, -3, -2, -1, 0\}$,

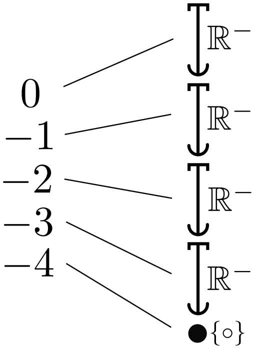
$$(Ł_5; \wedge, \vee, +, 0),$$

and the tomonoid $(\mathbb{R}^-; \wedge, \vee, +, 0)$.



Our aim is to coextend the tomonoid $Ł_5$ by $\mathbb{R}^-$.

We assume the result to be a tomonoid L ordered as follows:



The above example reversed

- We have to make $\mathbb{R}^-$
  a weakly transitive $\mathbb{R}^-$-module.
  We can set

$$f \star r \;=\; f + r \quad \text{where } f, r \in \mathbb{R}^-.$$

The above example reversed

- We have to make $\mathbb{R}^-$
  a weakly transitive $\mathbb{R}^-$-module.
  We can set

$$f \star r \;=\; f + r \quad \text{where } f, r \in \mathbb{R}^-.$$

- The tensor product is $\mathbb{R}^- \otimes_{\mathbb{R}^-} \mathbb{R}^- \cong \mathbb{R}^-$,
  and thus the homomorphisms are,
  for some $\sigma \in \mathbb{R}^-$, of the form

$$\varphi(r \otimes s) \;=\; r + s + \sigma.$$

The above example reversed

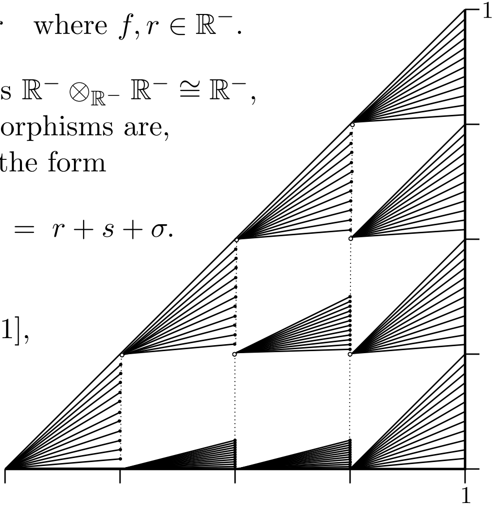
- We have to make $\mathbb{R}^-$
  a weakly transitive $\mathbb{R}^-$-module.
  We can set

$$f \star r \; = \; f + r \quad \text{where } f, r \in \mathbb{R}^-.$$

- The tensor product is $\mathbb{R}^- \otimes_{\mathbb{R}^-} \mathbb{R}^- \cong \mathbb{R}^-$,
  and thus the homomorphisms are,
  for some $\sigma \in \mathbb{R}^-$, of the form

$$\varphi(r \otimes s) \; = \; r + s + \sigma.$$

We conclude that our
coextension, scaled to $[0, 1]$,
is of this form:



Archimedean real coextensions

An integral tomonoid is Archimedean if, for any $a \leqslant b < 1$, there is an $n \geqslant 1$ such that $b^n \leqslant a$.

Definition

A coextension of an integral tomonoid by a filter F is called

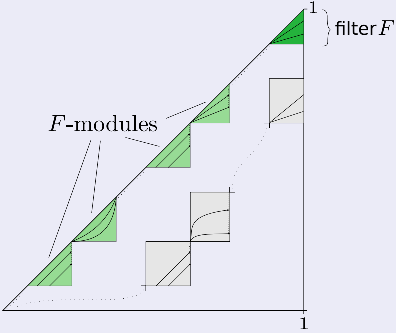
- Archimedean if F is Archimedean,
- real if each congruence class is order-isomorphic to a real interval.

Archimedean real coextensions

Theorem (Th. V., 2014) – qualitative formulation

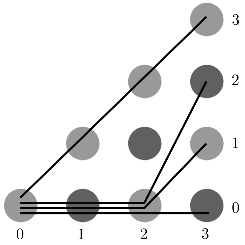
Let L be an Archimedean real coextension of the integral, quantic tomonoid P.

Given the congruence classes, we have, up to isomorphisms:



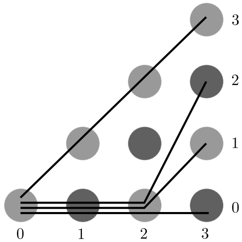
- The filter F is uniquely determined;
- up to one point, the F-modules are uniquely determined;
- up to one point per field, the remaining parts of L are uniquely determined.

Example of an Archimedean real coextension

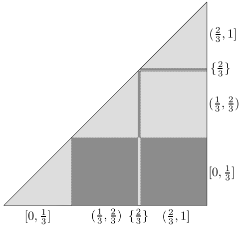


The tomonoid to be extended.

Example of an Archimedean real coextension

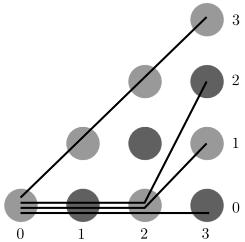


The tomonoid to be extended.

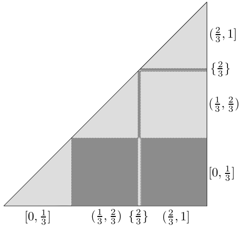


The congruence classes are real intervals.

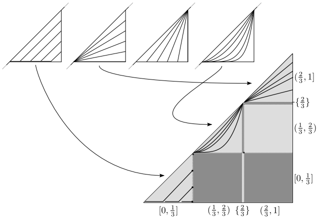
Example of an Archimedean real coextension



The tomonoid to be extended.

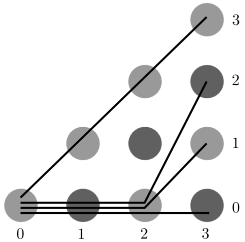


The congruence classes are real intervals.

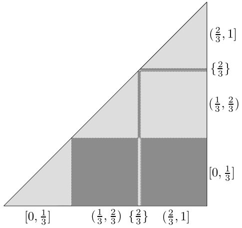


The unique choices for each "triangular part".

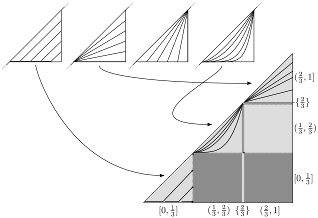
Example of an Archimedean real coextension



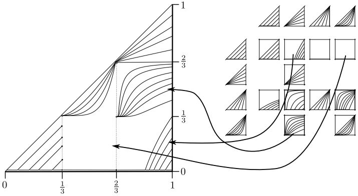
The tomonoid to be extended.



The congruence classes are real intervals.



The unique choices for each "triangular part".



The "rectangular parts" are uniquely determined as well.

H-transforms

Let us coextend the cancellative integral tomonoid P:

$$a \cdot c = b \cdot c \quad \text{implies} \quad a = b.$$

We put $L = \{(p, f) \colon p \in P,\ f \in F\}$,
endowed with the lexicographic order.

H-transforms

Let us coextend the cancellative integral tomonoid P:

$$a \cdot c = b \cdot c \quad \text{implies} \quad a = b.$$

We put $L = \{(p, f) \colon p \in P,\ f \in F\}$,
endowed with the lexicographic order.

- For each $p \in P \backslash \{1\}$,
  $\{(p, f) \colon f \in F\}$ is an F-module:

$$f \star (p, g) \;=\; (p, f \cdot g).$$

H-transforms

Let us coextend the cancellative integral tomonoid P:

$$a \cdot c = b \cdot c \quad \text{implies} \quad a = b.$$

We put $L = \{(p, f) \colon p \in P,\ f \in F\}$,
endowed with the lexicographic order.

- For each $p \in P \backslash \{1\}$,
  $\{(p, f) \colon f \in F\}$ is an F-module:

  $$f \star (p, g) \ = \ (p, f \cdot g).$$

- For each $p, q \in P \backslash \{1\}$, we have the bihomomorphism

  $$\psi((p, f), (q, g)) \ = \ (p \cdot q, f \cdot g).$$

H-transforms

Let us coextend the cancellative integral tomonoid P:

$$a \cdot c = b \cdot c \quad \text{implies} \quad a = b.$$

We put $L = \{(p, f) \colon p \in P,\ f \in F\}$,
endowed with the lexicographic order.

- For each $p \in P \backslash \{1\}$,
  $\{(p, f) \colon f \in F\}$ is an F-module:

  $$f \star (p, g) \;=\; (p, f \cdot g).$$

- For each $p, q \in P \backslash \{1\}$, we have the bihomomorphism

  $$\psi((p, f), (q, g)) \;=\; (p \cdot q, f \cdot g).$$

Thus we get the tomonoid L with

$$(p, f) \cdot (q, g) \;=\; (p \cdot q, f \cdot g).$$

Applied to tomonoids based on t-norms,
this is Zemánková's H-transform.

Dual of an F-module (D. Kruml)

Let the lattice R be an F-module for some integral ℓ-monoid F.

Assume that $f \star \_$ is residuated for each $f \in F$, that is, there is mapping $\backslash_\star$ such that

$$f \star r \leqslant s \quad \text{iff} \quad r \leqslant f \backslash_\star s.$$

Dual of an F-module (D. Kruml)

Let the lattice R be an F-module for some integral ℓ-monoid F.

Assume that $f \star \_$ is residuated for each $f \in F$, that is, there is mapping $\backslash_\star$ such that

$$f \star r \leqslant s \quad \text{iff} \quad r \leqslant f \backslash_\star s.$$

Then the dual lattice R^{op} together with

$$\backslash_\star \colon F \times R^{\mathrm{op}} \to R^{\mathrm{op}}$$

is an F-module as well, called the dual F-module.

The rotation

Let us coextend the three-element Łukasiewicz chain $Ł_3 = \{-2, -1, 0\}$, by an integral tomonoid F.

We put $L = \{(-2, f)\colon f \in F^{\mathrm{op}}\} \cup \{(-1, 0)\} \cup \{(0, f)\colon f \in F\}$, endowed with the lexicographic order.

The rotation

Let us coextend the three-element Łukasiewicz chain $Ł_3 = \{-2, -1, 0\}$, by an integral tomonoid F.

We put $L = \{(-2, f) \colon f \in F^{\text{op}}\} \cup \{(-1, 0)\} \cup \{(0, f) \colon f \in F\}$, endowed with the lexicographic order.

- We make $\{(-2, f) \colon f \in F^{\text{op}}\}$ into an F-module:

$$f \star (-2, g) \;=\; (-2, f \backslash_{\star} g).$$

  The singleton $\{(-1, 0)\}$ is a trivial F-module.
- The only bihomomorphism to be determined is trivial.

The rotation

Let us coextend the three-element Łukasiewicz chain $Ł_3 = \{-2, -1, 0\}$, by an integral tomonoid F.

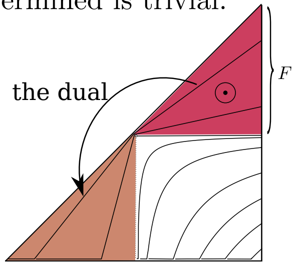
We put $L = \{(-2, f) \colon f \in F^{\mathrm{op}}\} \cup \{(-1, 0)\} \cup \{(0, f) \colon f \in F\}$, endowed with the lexicographic order.

- We make $\{(-2, f) \colon f \in F^{\mathrm{op}}\}$ into an F-module:

  $$f \star (-2, g) \;=\; (-2, f \setminus_{\star} g).$$

  The singleton $\{(-1, 0)\}$ is a trivial F-module.
- The only bihomomorphism to be determined is trivial.

Applied to t-norms, this is Jenei's rotation construction.



Grillet's and Leech's group coextensions of monoids generalise Schreier's theory of group extensions.

Grillet's and Leech's group coextensions of monoids generalise Schreier's theory of group extensions.

The construction can be adapted to the ordered case: pomonoid coextension of pomonoids.

Group extensions in an ordered setting (J. Janda, Th. V.)

Grillet's and Leech's group coextensions of monoids generalise Schreier's theory of group extensions.

The construction can be adapted to the ordered case: pomonoid coextension of pomonoids.

Provided that the congruence is filter-induced, this construction is covered in our framework as well.

Rees congruences

From now on, commutativity is no longer necessary.

Lemma

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and $q \in L$.
Define, for $a, b \in L$,

$$a \, \theta_q \, b \quad \text{if } a = b \text{ or } a, b \leqslant q.$$

Then θ_q is a congruence.

Rees congruences

From now on, commutativity is no longer necessary.

Lemma

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral ℓ-monoid and $q \in L$.
Define, for $a, b \in L$,

$$a \, \theta_q \, b \quad \text{if } a = b \text{ or } a, b \leqslant q.$$

Then θ_q is a congruence.

We denote the quotient by L/q and
we call it the Rees quotient by q.

Example

Consider the five-element Łukasiewicz chain
$Ł_5 = \{-4, -3, -2, -1, 0\}$,

$$(Ł_5; \wedge, \vee, +, 0),$$

and the atom of $Ł_5$, i.e., -3.

Then the Rees quotient of $Ł_5$ by -3 is

$$Ł_4 = \{-3, -2, -1, 0\}.$$

Definition

Let L be a finite integral tomonoid and
let $P = L/\alpha$ be the Rees quotient by the atom α of L.
Then we call L a one-element (Rees) coextension of P.

One-element coextensions (M. Petrík, Th. V.)

Definition

Let L be a finite integral tomonoid and
let $P = L/\alpha$ be the Rees quotient by the atom α of L.
Then we call L a one-element (Rees) coextension of P.

Challenge

Given a finite integral tomonoid P,
determine all one-element coextensions of P.

Tomonoid partitions

Definition

Let $(L; \wedge, \vee, \cdot, 1)$ be an integral tomonoid.

For $(a, b), (c, d) \in L \times L$, let

$$(a, b) \sim (c, d) \quad \text{if} \quad a \cdot b = c \cdot d,$$
$$(a, b) \trianglelefteq (c, d) \quad \text{if } a \leqslant c \text{ and } b \leqslant d.$$

Then we call $(L^2, \trianglelefteq, \sim, (1, 1))$ a tomonoid partition.

0	t	u	v	w	x	y	z	1	
0	t	u	v	w	x	y	z	1	1
0	0	t	u	v	w	w	x	z	z
0	0	0	t	u	v	v	w	y	y
0	0	0	t	u	v	v	w	x	x
0	0	0	0	t	u	u	v	w	w
0	0	0	0	0	t	t	u	v	v
0	0	0	0	0	0	0	t	u	u
0	0	0	0	0	0	0	0	t	t
0	0	0	0	0	0	0	0	0	0

Tomonoid partitions

Lemma

Let $(L; \leqslant)$ be a chain, let $1 \in L$,
and let $\sim$ be an equivalence relation on L^2.

Then $(L^2, \trianglelefteq, \sim, (1,1))$ is a tomonoid partition if and only if:

(P1) $(a,b) \sim (a',b') \trianglelefteq (c,d) \sim (c',d') \trianglelefteq (a,b)$ implies $(a,b) \sim (c,d)$.

(P2) For any $a, b \in L$ there is exactly one $c \in L$ such that $(a,b) \sim (1,c) \sim (c,1)$.

(P3) $(a,b) \sim (d,1)$ and $(b,c) \sim (1,e)$ imply $(d,c) \sim (a,e)$.

Conditions (P1), (P2)

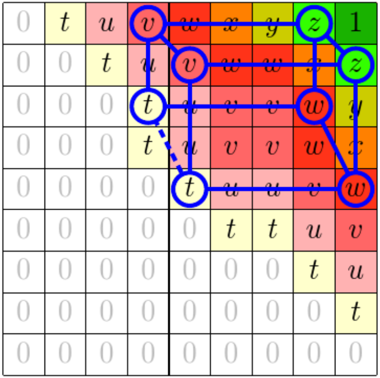
(P1) $(a, b) \sim (a', b') \trianglelefteq (c, d) \sim (c', d') \trianglelefteq (a, b)$ implies $(a, b) \sim (c, d)$.

(P2) For any $a, b \in L$ there is exactly one $c \in L$ such that $(a, b) \sim (1, c) \sim (c, 1)$.

0	t	u	v	w	x	y	z	1
0	0	t	u	v	w	w	x	z
0	0	0	t	u	v	v	w	y
0	0	0	t	u	v	v	w	x
0	0	0	0	t	u	u	v	w
0	0	0	0	0	t	t	u	v
0	0	0	0	0	0	0	t	u
0	0	0	0	0	0	0	0	t
0	0	0	0	0	0	0	0	0

The "Reidemeister" condition (P3)

(P3) $(a, b) \sim (d, 1)$ and
$(b, c) \sim (1, e)$ imply
$(d, c) \sim (a, e)$.



The Rees quotient by the atom

0	t	u	v	w	x	y	z	1
0	0	t	u	v	w	w	x	z
0	0	0	t	u	v	v	w	y
0	0	0	t	u	v	v	w	x
0	0	0	0	t	u	u	v	w
0	0	0	0	0	t	t	u	v
0	0	0	0	0	0	0	t	u
0	0	0	0	0	0	0	0	t
0	0	0	0	0	0	0	0	0

0	u	v	w	x	y	z	1
0	0	u	v	w	w	x	z
0	0	0	u	v	v	w	y
0	0	0	u	v	v	w	x
0	0	0	0	u	u	v	w
0	0	0	0	0	0	u	v
0	0	0	0	0	0	0	u
0	0	0	0	0	0	0	0

Now let us go the other way round ...

The one-element coextensions

We determine the Archimedean one-element coextensions of the six-element tomonoid shown.

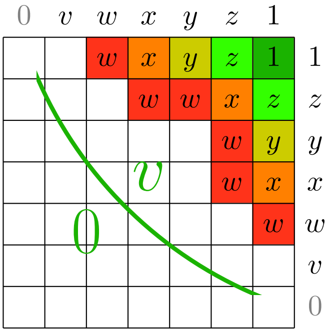
	0	w	x	y	z	1
1	0	w	x	y	z	1
z	0	0	w	w	x	z
y	0	0	0	0	w	y
x	0	0	0	0	w	x
w	0	0	0	0	0	w
0	0	0	0	0	0	0

The one-element coextensions

0	v	w	x	y	z	1	
		w	x	y	z	1	1
			w	w	x	z	z
					w	y	y
					w	x	x
						w	w
							v
							0

- We insert a row and a column for a new atom.

The one-element coextensions



- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
			w	w	x	z	z
					w	y	y
					w	x	x
						w	w
						v	v
						0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
	0		w	w	x	z	z
					w	y	y
					w	x	x
						w	w
					0	v	v
						0	0

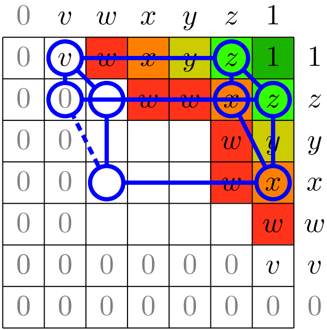
- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0				w	y	y
0	0				w	x	x
0	0					w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.

The one-element coextensions



- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0				w	y	y
0	0	0			w	x	x
0	0	0				w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0				w	y	y
0	0	0			w	x	x
0	0	0	0			w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0				w	y	y
0	0	0			w	x	x
0	0	0	0	0		w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0				w	y	y
0	0	0			w	x	x
0	0	0	0	0		w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

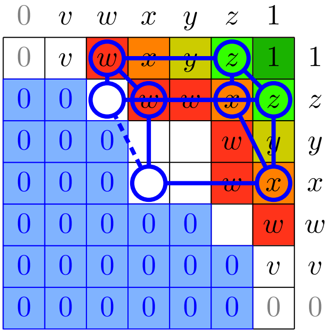
- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0	0			w	y	y
0	0	0			w	x	x
0	0	0	0	0		w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions



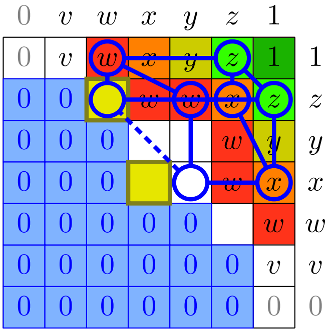
- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0	0			w	y	y
0	0	0			w	x	x
0	0	0	0	0		w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions



- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0	0			w	y	y
0	0	0			w	x	x
0	0	0	0	0		w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0	0			w	y	y
0	0	0			w	x	x
0	0	0	0	0		w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0	0	w	w	x	z	z
0	0	0	0	0	w	y	y
0	0	0	0	0	w	x	x
0	0	0	0	0	0	w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).
- We reduce the partition to two subsets, taking into account monotonicity.

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0	0	w	w	x	z	z
0	0	0	0	v	w	y	y
0	0	0	0	0	w	x	x
0	0	0	0	0	0	w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).
- We reduce the partition to two subsets, taking into account monotonicity.

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0	v	w	w	x	z	z
0	0	0	v	v	w	y	y
0	0	0	v	v	w	x	x
0	0	0	0	0	v	w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).
- We reduce the partition to two subsets, taking into account monotonicity.

The one-element coextensions

0	v	w	x	y	z	1	
0	v	w	x	y	z	1	1
0	0		w	w	x	z	z
0	0	0			w	y	y
0	0	0			w	x	x
0	0	0	0	0		w	w
0	0	0	0	0	0	v	v
0	0	0	0	0	0	0	0

- We insert a row and a column for a new atom.
- The lower area must now be divided between the new atom and the new 0.
- We identify the clear cases.
- We identify pairs equivalent due to conditions (P1)–(P3).
- We reduce the partition to two subsets, taking into account monotonicity.

Theorem (M. Petrík, Th. V.)

In the way shown, we obtain all one-element coextensions of a given finite, integral tomonoid.